Asian Journal of Probability and Statistics

Volume 26, Issue 10, Page 171-191, 2024; Article no.AJPAS.122335

ISSN: 2582-0230

The Two Component Generalized Finite Erlang Mixture and its Properties and Special Cases

Beatrice M. Gathongo [a*]

[a] *Department of Mathematics, University of Nairobi, Kenya.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: https://doi.org/10.9734/ajpas/2024/v26i10666

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/122335

Received: 05/07/2024
Accepted: 10/09/2024
Published: 09/10/2024

Original Research Article

Abstract

This research proposes the construction of a two-component generalized finite Erlang mixture with four parameters. Three distinct cases of this mixture are presented, differing in their mixing weights and corresponding component probability (Erlang) distributions. Special cases of these mixtures, including the one-, two-, and three-parameter finite Erlang mixtures, have also been derived. The statistical properties examined for these mixtures include the distribution function, survival function, hazard function, moment generating function, raw and central moments, mean, variance, coefficient of skewness, coefficient of kurtosis, and order statistics. Parameter estimation for the finite Erlang mixtures was conducted using both the method of moments and maximum likelihood estimation. Furthermore, the 4-parameter mixed Erlang distributions were applied to a real dataset on the relief times of patients receiving an analgesic, to evaluate their goodness of fit. The results demonstrate the potential of these mixtures to provide robust modeling for empirical data, suggesting their applicability in various statistical and practical contexts.

**Corresponding author: E-mail: beatricegathongo@gmail.com;*

Cite as: *Gathongo, Beatrice M. 2024. "The Two Component Generalized Finite Erlang Mixture and Its Properties and Special Cases". Asian Journal of Probability and Statistics 26 (10):171-91. https://doi.org/10.9734/ajpas/2024/v26i10666.*

Keywords: Finite Erlang mixture; generalization; component; mixing weight; parameter.

1 Introduction

The gamma distribution is a versatile probability density function that is widely applicable, particularly in modeling the time between events in stochastic processes. It is defined by two parameters: shape and rate (or scale). The flexibility of the gamma distribution lies in its shape parameter, which allows it to encompass several other distributions as special cases. Notably, when the shape parameter equals 1, the gamma distribution simplifies to the exponential distribution. When the shape parameter is an integer, it becomes the Erlang distribution, a specific case of the gamma distribution that is often preferred for its practical applications owing to its discretized shape parameter. The Erlang distribution is especially useful in modeling queuing systems and other stochastic processes. Generalization involves expanding a distribution by introducing additional parameters, thereby broadening its applicability and enhancing its modeling capacity. This approach enables the development of new distributions that can capture a wider range of behaviors and characteristics in data, making them powerful tools in statistical modeling and analysis.

When there are two or more sub-populations in a population, mixed distributions can be used to model the population, such that components of the mixture will suit the sub-populations. Thus, mixture distributions are used to model data that the basic distributions may fail to fit. Pearson[1] introduced mixed distributions in 1894 when he constructed a finite mixture using two normal distributions with different means and variances. Finite mixtures are among the three types of mixtures, the other two being continuous and discrete. Finite mixtures are defined by the number of components which can be ≥ 2, and the number of parameters which can be at least one. Lindley[2] introduced finite Erlang mixtures when he presented the Lindley distribution, a two-component mixture, which he applied in studying fiducial distributions and Bayes' theorem. There since have been innumerable work on finite Erlang mixtures. Shanker et al.[3], Shanker[4], Shanker et al.[5], Shanker et al.[6], Mussie and Shanker[7], Tesfay and Shanker[8], Tesfay and Shanker[9], and Nwikpe and Iwok[10] are among authors who studied three component finite Erlang mixtures, while Shanker[11], Shanker[12] and Rashid et al.[13] presented four component finite Erlang mixtures. Shanker[14] and Shanker[15] defined five and six-component finite Erlang mixtures respectively.

The focus of this paper is on two component four parameter generalized finite Erlang mixtures, and their special cases, the one, two, and three-parameter finite Erlang mixtures. Zakerzadeh and Dolati[16] defined the generalized Lindley distribution with three parameters as a mixture of gamma(α,θ) and gamma($\alpha+1,\theta$), with respective mixing weights $\frac{\theta}{\theta+\gamma}$ and $\frac{\gamma}{\theta+\gamma}$. Nadarajah et al.[17] demonstrated that the cdf of the two-parameter generalized Lindley distribution they proposed, was the α^{th} order statistic of the Lindley distribution. A two-parameter Lindley distribution was studied by Shanker et al.[18] as the sum of $\frac{\theta}{\theta+\alpha}$ gamma$(1,\theta)$ and $\frac{\alpha}{\theta+\alpha}$ gamma$(2,\theta)$, and Shanker and Mishra[19] presented it as the sum of $\frac{\alpha\theta}{\alpha\theta+1}$gamma$(1,\theta)$ and $\frac{1}{\alpha\theta+1}$gamma$(2,\theta)$. Shanker and Mishra[20] defined a Quasi Lindley distribution with two parameters as $\frac{\alpha}{\alpha+1}$gamma$(1,\theta)$+$\frac{1}{\alpha+1}$ gamma$(2,\theta)$, and Shanker and Amanuel[21] expressed a new Quasi Lindley distribution with two parameters as $\frac{\theta^2}{\theta^2+\alpha}$gamma$(1,\theta)$+$\frac{\alpha}{\theta^2+\alpha}$gamma$(2,\theta)$. The cdf of the Transmuted Lindley distribution with two parameters was expressed by Merovci[22] as the sum of the cdf the Lindley distribution and the squared cdf of the Lindley distribution, with respective weights $(1+\lambda)$ and $-\lambda$. They also presented the Lindley distribution as a special case of the Transmuted Lindley distribution. Elbatal et al.[23] proposed a new generalized Lindley distribution with three parameters as $\frac{\theta}{\theta+1}$gamma(α,θ)+$\frac{1}{\theta+1}$gamma(β,θ). The Lindley, gamma, and exponential distributions were expressed as special cases of the distribution.

The cdf of a four-parameter beta-generalized Lindley (BGL) distribution was expressed in terms of the cdf of the generalized Lindley distribution by Oluyede[24]. A new generalized Lindley distribution with five parameters was presented by Abouammoh[25] as the sum of gamma(τ,θ) and gamma(η,θ), with respective mixing weights $\frac{\theta^\rho}{\gamma+\theta^\rho}$ and $\frac{\gamma}{\gamma+\theta^\rho}$. Shanker[26] proposed a one parameter Shanker distribution as $\frac{\theta^2}{\theta^2+1}$gamma$(1,\theta)$+$\frac{1}{\theta^2+1}$gamma$(2,\theta)$.

The Akash distribution with one parameter, $\frac{\theta^2}{\theta^2+2}\text{gamma}(1,\theta)+\frac{2}{\theta^2+2}\text{gamma}(3,\theta)$, was presented by Shanker[27]. Shanker[28] studied a one parameter Rama distribution, $\frac{\theta^3}{\theta^3+6}\text{gamma}(1,\theta)+\ \frac{6}{\theta^3+6}\text{gamma}(4,\theta)$. Shanker[29] defined a one parameter Suja distribution as $\frac{\theta^4}{\theta^4+24}\text{gamma}(1,\theta)\ +\frac{24}{\theta^4+24}\text{gamma}(5,\theta)$.

The structure of this research is outlined as follows: The mixed Erlang distribution and its properties have been presented in section 2. The three cases of the two-component four parameter generalized finite Erlang mixture, and their properties and special cases have been presented in sections 3, 4, and 5. In section 6, the three cases of the generalized finite Erlang mixture have been fitted to a data set alongside other distributions to assess their goodness of fit, and the conclusion of the paper is provided in section 7.

2 The Two-Component Finite Erlang Mixture and its Properties

- The gamma (α,θ) distribution is given by;

$$f(x;\alpha,\theta)=\frac{\theta^\alpha}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1},\quad x>0;\alpha>0,\theta>0 \tag{2.1}$$

- The exponential (θ) distribution,

$$f(x;\theta)=\theta e^{-\theta x},\quad x>0;\theta>0 \tag{2.2}$$

  is a special case of the gamma distribution when $\alpha=1$.

- The Erlang (n,θ) distribution,

$$f(x;n,\theta)=\frac{\theta^n}{\Gamma(n)}e^{-\theta x}x^{n-1},\quad x>0;\theta>0,n=1,2,3,... \tag{2.3}$$

  is a special case of the gamma distribution when $\alpha=n$ is a positive integer.

Properties of the Erlang (n,θ) distribution

1. The distribution function (CDF) of the Erlang distribution is given by

$$F(x;n,\theta)=1-e^{-\theta x}\sum_{t=0}^{n-1}\frac{(\theta x)^t}{t!}=\frac{\gamma(n,\theta x)}{\Gamma(n)} \tag{2.4}$$

2. Its survival and hazard functions are, respectively,

$$S(x;n,\theta)=1-F(x;n,\theta)=e^{-\theta x}\sum_{t=0}^{n-1}\frac{(\theta x)^t}{t!}=\frac{\Gamma(n,\theta x)}{\Gamma(n)}$$
$$\text{and}\quad h(x;n,\theta)=\frac{f(x;n,\theta)}{S(x;n,\theta)}=\frac{\theta^n}{\Gamma(n,\theta x)}e^{-\theta x}x^{n-1} \tag{2.5}$$

3. The moment generating function (MGF) is

$$M_x(t)=E(e^{tx})=\frac{\theta^n}{\Gamma(n)}\int_0^\infty e^{-(\theta-t)x}x^{n-1}dx=\left(\frac{\theta}{\theta-t}\right)^n \tag{2.6}$$

4. The r^{th} raw moment of the Erlang distribution is given by

$$E(X^r)=\frac{\theta^n}{\Gamma(n)}\int_0^\infty e^{-\theta x}x^{r+n-1}dx=\frac{\Gamma(r+n)}{\theta^r\Gamma(n)} \tag{2.7}$$

and hence, the first four raw moments are

$$E(X)=\frac{n}{\theta},\quad E(X^2)=\frac{n(n+1)}{\theta^2},\quad E(X^3)=\frac{n(n+1)(n+2)}{\theta^3},$$
$$\text{and}\quad E(X^4)=\frac{n(n+1)(n+2)(n+3)}{\theta^4} \tag{2.8}$$

Remark: The above properties of the Erlang distribution will be applied in obtaining properties of the finite Erlang mixtures, since the component probability distributions of the Erlang mixtures are Erlang distributions.

5. The central moments are thus
   i. Variance
$$Var(X)=E(X^2)-[E(X)]^2=\frac{n(n+1)}{\theta^2}-\frac{n^2}{\theta^2}=\frac{n}{\theta^2} \tag{2.9}$$
   ii. Coefficient of skewness
$$\frac{\mu_3}{\sigma^3}=\frac{1}{\sigma^3}\left\{E(X^3)-3E(X)E(X^2)+2[E(X)]^3\right\}=\frac{1}{\sigma^3}\left\{\frac{n(n+1)(n+2)}{\theta^3}-\frac{3n^2(n+1)}{\theta^3}+\frac{3n^3}{\theta^3}\right\} \tag{2.10}$$
   iii. Coefficient of kurtosis
$$\frac{\mu_4}{\sigma^4}=\frac{1}{\sigma^4}\left\{E(X^4)-4[E(X)][E(X^3)]+6[E(X)]^2[E(X^2)]-3[E(X)]^4\right\}$$
$$=\frac{1}{\sigma^4}\left\{\frac{n(n+1)(n+2)(n+3)}{\theta^4}-\frac{4n^2(n+1)(n+2)}{\theta^4}+\frac{6n^3(n+1)}{\theta^4}-\frac{3n^4}{\theta^4}\right\} \tag{2.11}$$
6. The probability function of the k^{th} order statistic is
$$f_k(x)=\frac{n!f(x)}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\,[F(x)]^{i+k-1}$$
$$=\frac{n!\theta^n e^{-\theta x}x^{n-1}}{(k-1)!(n-k)!\Gamma(n)}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\left[\frac{\gamma(n,\theta x)}{\Gamma(n)}\right]^{i+k-1} \tag{2.12}$$
7. The method of moments estimators for the Erlang distribution parameters are given by
$$\hat{n}=\frac{n\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2},\quad \text{and}\quad \hat{\theta}=\frac{n\bar{x}}{\sum_{i=1}^{n}(x_i-\bar{x})^2} \tag{2.13}$$
   and the maximum likelihood estimators are
$$\hat{\theta}=\frac{\hat{n}}{\bar{x}},\quad \text{and}\quad \frac{\delta}{\delta\hat{n}}\text{log}\Gamma(\hat{n})-\text{log}\hat{n}+\text{log}\bar{x}-\frac{1}{n}\sum_{i=1}^{n}\text{log}x_i=0 \tag{2.14}$$

- A k-component finite mixture $f(x)$ is given by;
$$f(x)=\sum_{j=1}^{k}\omega_j f_j(x) \tag{2.15}$$
  where $f_j(x)$ is the j^{th} component probability distribution which can be discrete or continuous, and ω_j is the weighted j^{th} mixing weight for $j=1,2,3,...,k$, with the following properties; $\omega_j>0$ and $\sum_{j=1}^{k}\omega_j=1$. Hence, $f_j(x)$ is given by (2.3) in (2.15) for a k-component finite Erlang mixture, and for two components, the mixtures will thus be of the form;
$$g(x)=\omega g_1(x)+(1-\omega)g_2(x) \tag{2.16}$$
  where $g_i(x)$ is an Erlang distribution.

2.1 Properties of the k-component finite mixture

1. The distribution function (CDF) of the finite mixture can be expressed as,

$$
\begin{aligned}
F(x) &= \int_0^x f(t)dt = \sum_{j=1}^{k} \omega_j \int_0^x f_j(t)dt = \sum_{j=1}^{k} \omega_j F_j(x) \\
&= \sum_{t=0}^{x} f(t)dt = \sum_{j=1}^{k} \omega_j \sum_{t=0}^{x} f_j(t)dt = \sum_{j=1}^{k} \omega_j F_j(x) \qquad (2.17)
\end{aligned}
$$

   where $F_j(x)$'s are the respective continuous and discrete j^{th} component CDFs.

2. The survival function is given by

$$
\begin{aligned}
S(x) &= \int_x^\infty f(t)dt = \sum_{j=1}^{k} \omega_j \int_x^\infty f_j(t)dt = \sum_{j=1}^{k} \omega_j S_j(x) \\
&= \sum_{t=x}^{\infty} f(t)dt = \sum_{j=1}^{k} \omega_j \sum_{t=x}^{\infty} f_j(t)dt = \sum_{j=1}^{k} \omega_j S_j(x) \qquad (2.18)
\end{aligned}
$$

   where $S_j(x)$'s are the continuous and discrete j^{th} component survival functions respectively.

3. The hazard function of the mixture is given by

$$
h(x) = \frac{f(x)}{S(x)} \qquad (2.19)
$$

4. The moment generating function (MGF) of the finite mixture is

$$
\begin{aligned}
M_x(t) = E(e^{tx}) &= \int_0^\infty e^{tx} f(x)dx = \sum_{j=1}^{k} \omega_j \int_0^\infty e^{tx} f_j(x)dx = \sum_{j=1}^{k} \omega_j M_x(t)_j \\
&= \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{j=1}^{k} \omega_j \sum_{x=0}^{\infty} e^{tx} f_j(x) = \sum_{j=1}^{k} \omega_j M_x(t)_j \qquad (2.20)
\end{aligned}
$$

   for the continuous and discrete component distributions respectively, where $M_x(t)_j$ is the j^{th} component MGF.

5. The r^{th} raw moment of the mixture is given by

$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^r f(x)dx = \sum_{j=1}^{k} \omega_j \int_0^\infty x^r f_j(x)dx = \sum_{j=1}^{k} \omega_j E_j(X^r) \\
&= \sum_{x=0}^{\infty} x^r f(x) = \sum_{j=1}^{k} \omega_j \sum_{x=0}^{\infty} x^r f_j(x) = \sum_{j=1}^{k} \omega_j E_j(X^r) \qquad (2.21)
\end{aligned}
$$

   for the continuous and discrete component distributions respectively, where $E_j(X^r)$ is the r^{th} raw moment of the j^{th} component.
   The first four raw moments of the mixture will thus be

$$
E(X) = \sum_{j=1}^{k} \omega_j E_j(X), \quad E(X^2) = \sum_{j=1}^{k} \omega_j E_j(X^2), \quad E(X^3) = \sum_{j=1}^{k} \omega_j E_j(X^3),
$$

$$
\text{and} \quad E(X^4) = \sum_{j=1}^{k} \omega_j E_j(X^4) \qquad (2.22)
$$

6. The central moments of the mixed distribution are

   i. Variance

$$\mu_2 = E(X^2) - [E(X)]^2 = \sum_{j=1}^{k} \omega_j E_j(X^2) - \left[\sum_{j=1}^{k} \omega_j E_j(X)\right]^2 \tag{2.23}$$

   ii. Coefficient of skewness

$$\begin{aligned}\frac{\mu_3}{\sigma^3} &= \frac{1}{\sigma^3}\left\{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3\right\}\\ &= \frac{1}{\sigma^3}\left\{\sum_{j=1}^{k} \omega_j E_j(X^3) - 3\left[\sum_{j=1}^{k} \omega_j E_j(X)\right]\left[\sum_{j=1}^{k} \omega_j E_j(X^2)\right] + 2\left[\sum_{j=1}^{k} \omega_j E_j(X)\right]^3\right\}\end{aligned} \tag{2.24}$$

   iii. Coefficient of kurtosis

$$\begin{aligned}\frac{\mu_4}{\sigma^4} &= \frac{1}{\sigma^4}\left\{E(X^4) - 4[E(X)][E(X^3)] + 6[E(X)]^2[E(X^2)] - 3[E(X)]^4\right\}\\ &= \frac{1}{\sigma^4}\left\{\left[\sum_{j=1}^{k} \omega_j E_j(X^4)\right] - 4\left[\sum_{j=1}^{k} \omega_j E_j(X)\right]\left[\sum_{j=1}^{k} \omega_j E_j(X^3)\right] + 6\left[\sum_{j=1}^{k} \omega_j E_j(X)\right]^2\right.\\ &\quad \left.\left[\sum_{j=1}^{k} \omega_j E_j(X^2)\right] - 3\left[\sum_{j=1}^{k} \omega_j E_j(X)\right]^4\right\}\end{aligned} \tag{2.25}$$

7. The probability function of the k^{th} order statistic for the finite mixture is given by

$$\begin{aligned}f_k(x) &= \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x)\\ &= \frac{n!f(x)}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i[F(x)]^{i+k-1}\\ &= \frac{n!f(x)}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\left[\sum_{j=1}^{k}\omega_j F_j(x)\right]^{i+k-1}\end{aligned} \tag{2.26}$$

   where $f(x)$ is the mixture distribution and $F_j(x)$ is the j^{th} component distribution function in the mixed distribution.

8. Parameter estimation

   Method of moments estimation (MME): The r^{th} sample moments have been equated to the r^{th} moments of the mixed distributions to obtain parameter estimates of the finite mixtures, that is,

$$\frac{1}{n}\sum_{i=1}^{n} = x_i^r = E(X^r), \quad r = 1, 2, 3, ... \tag{2.27}$$

   Maximum likelihood estimation (MLE): The parameters of the finite mixtures have also been estimated using MLE. The log-likelihood functions obtained from the likelihood functions have been differentiated with respect to the parameters, and the resulting equations solved simultaneously to obtain estimates of the parameters.

3 The Two Component four Parameter Finite Erlang Mixture: Case 1

$$\text{Let}\quad g_1(x) = \mathsf{Gamma}(\alpha,\theta),\quad g_2(x) = \mathsf{Gamma}(\alpha+1,\theta),\quad \omega = \frac{\theta}{\theta+s}\quad \text{and}\quad (1-\omega) = \frac{s}{\theta+s}$$

$$\begin{aligned}
\text{Then,}\quad g(x) &= \frac{\theta}{\theta+s}\frac{\theta^{\alpha}}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1} + \frac{s}{\theta+s}\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}e^{-\theta x}x^{\alpha} = \frac{\theta^{\alpha+1}}{\theta+s}e^{-\theta x}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{sx^{\alpha}}{\Gamma(\alpha+1)}\right)\\
&= \frac{\theta^{\alpha+1}}{\theta+s}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}e^{-\theta x}\left(\alpha+sx\right)\\
\text{let}\quad s &= \frac{\delta}{\beta}\\
g(x) &= \frac{\theta^{\alpha+1}}{\theta+\frac{\delta}{\beta}}e^{-\theta x}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}\left(\alpha+\frac{\delta}{\beta}x\right)\\
&= \frac{\beta\alpha+\delta x}{\beta\theta+\delta}\frac{\theta^{\alpha+1}e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha+1)},\quad x>0;\theta>0,\beta>0,\delta>0,\alpha=1,2,3,... \qquad (3.1)
\end{aligned}$$

which is a 4-parameter generalized Lindley distribution with the following properties and special cases.

3.1 Properties

1. The CDF of the mixture is given by

$$\begin{aligned}
G(x) &= \frac{\theta}{\theta+s}\frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{s}{\theta+s}\frac{\gamma(\alpha+1,\theta x)}{\Gamma(\alpha+1)} = \frac{[\theta\alpha\gamma(\alpha,\theta x)+s\gamma(\alpha+1,\theta x)]}{(\theta+s)\Gamma(\alpha+1)}\\
&= \frac{[\theta\alpha\beta\gamma(\alpha,\theta x)+\delta\gamma(\alpha+1,\theta x)]}{(\beta\theta+\delta)\Gamma(\alpha+1)} \qquad (3.2)
\end{aligned}$$

2. The survival function is

$$\begin{aligned}
S(x) &= \frac{\theta}{\theta+s}\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{s}{\theta+s}\frac{\Gamma(\alpha+1,\theta x)}{\Gamma(\alpha+1)} = \frac{[\theta\alpha\Gamma(\alpha,\theta x)+s\Gamma(\alpha+1,\theta x)]}{(\theta+s)\Gamma(\alpha+1)}\\
&= \frac{[\theta\alpha\beta\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+1,\theta x)]}{(\beta\theta+\delta)\Gamma(\alpha+1)} \qquad (3.3)
\end{aligned}$$

3 and the hazard function is thus

$$h(x) = \frac{\frac{\beta\alpha+\delta x}{\beta\theta+\delta}\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}e^{-\theta x}x^{\alpha-1}}{\frac{[\theta\alpha\beta\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+1,\theta x)]}{(\beta\theta+\delta)\Gamma(\alpha+1)}} = \frac{(\beta\alpha+\delta x)\theta^{\alpha+1}e^{-\theta x}x^{\alpha-1}}{[\theta\alpha\beta\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+1,\theta x)]} \qquad (3.4)$$

4. The MGF of the mixed distribution is

$$M_x(t) = \frac{\theta}{\theta+s}\left(\frac{\theta}{\theta-t}\right)^{\alpha} + \frac{s}{\theta+s}\left(\frac{\theta}{\theta-t}\right)^{\alpha+1} = \frac{\theta^{\alpha+1}[\theta-t+s]}{(\theta+s)(\theta-t)^{\alpha+1}} = \frac{\theta^{\alpha+1}[\beta(\theta-t)+\delta]}{(\beta\theta+\delta)(\theta-t)^{\alpha+1}} \qquad (3.5)$$

5. The r^{th} moment about the origin of the 4-parameter generalized generalized Lindley distribution is given by

$$E(X^r) = \frac{\theta}{\theta+s}\frac{\Gamma(s+\alpha)}{\theta^r\Gamma(\alpha)} + \frac{s}{\theta+s}\frac{\Gamma(r+\alpha+1)}{\theta^r\Gamma(\alpha+1)} = \frac{[\alpha\theta+s(r+\alpha)]\,\Gamma(r+\alpha)}{(\theta+s)\theta^r\Gamma(\alpha+1)} = \frac{[\beta\alpha\theta+\delta(r+\alpha)]\,\Gamma(r+\alpha)}{(\beta\theta+\delta)\theta^r\Gamma(\alpha+1)} \qquad (3.6)$$

and the first four moments are therefore

$$E(X) = \frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta} \tag{3.7}$$

$$E(X^2) = \frac{[\beta\alpha\theta + \delta(2+\alpha)]\,(1+\alpha)}{(\beta\theta+\delta)\theta^2} \tag{3.8}$$

$$E(X^3) = \frac{[\beta\alpha\theta + \delta(3+\alpha)]\,(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^3} \tag{3.9}$$

$$E(X^4) = \frac{[\beta\alpha\theta + \delta(4+\alpha)]\,(3+\alpha)(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^4} \tag{3.10}$$

6. The central moments are

   i. Variance

$$\begin{aligned}\mu_2 &= \frac{[\beta\alpha\theta + \delta(2+\alpha)]\,(1+\alpha)}{(\beta\theta+\delta)\theta^2} - \left\{\frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta}\right\}^2 \\ &= \frac{\beta\alpha(\alpha+1)\theta(\beta\theta+\delta) + \delta(\alpha+1)(\alpha+2)(\beta\theta+\delta) - (\beta\alpha\theta)^2 - \delta^2(\alpha+1)^2 + 2\beta\alpha\delta\theta(\alpha+1)}{\theta^2(\beta\theta+\delta)^2}\end{aligned} \tag{3.11}$$

   ii. Coefficient of Skewness

$$\begin{aligned}\frac{\mu_3}{\sigma^3} &= \frac{1}{\sigma^3}\left\{\frac{[\beta\alpha\theta + \delta(3+\alpha)]\,(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^3} - 3\left[\frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta}\right]\left[\frac{[\beta\alpha\theta + \delta(2+\alpha)]\,(1+\alpha)}{(\beta\theta+\delta)\theta^2}\right] + \right. \\ &\quad \left. 2\left[\frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta}\right]^3\right\} \\ &= \frac{(\alpha+1)(\alpha+2)(\beta\theta+\delta)^2[\beta\alpha\theta + \delta(\alpha+3)] - 3(\alpha+1)(\beta\theta+\delta)[(\beta\alpha\theta)^2 + \beta\alpha\theta\delta(\alpha+1) +}{\sigma^3\theta^3(\beta\theta+\delta)^3} \\ &\quad \underline{\beta\alpha\theta\delta(\alpha+2) + \delta^2(\alpha+1)(\alpha+2)] + 2[(\beta\alpha\theta)^3 + 3(\beta\alpha\theta)^2\delta(\alpha+1) + 3\beta\alpha\theta\delta^2(\alpha+1)^2 + \delta^3(\alpha+1)^3]}\end{aligned} \tag{3.12}$$

   iii. Coefficient of Kurtosis

$$\begin{aligned}\frac{\mu_4}{\sigma^4} &= \frac{1}{\sigma^4}\left\{\frac{[\beta\alpha\theta + \delta(4+\alpha)]\,(3+\alpha)(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^4} - 4\left[\frac{[\beta\alpha\theta + \delta(3+\alpha)]\,(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^3}\right]\left[\frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta}\right] + \right. \\ &\quad \left. 6\left[\frac{[\beta\alpha\theta + \delta(2+\alpha)]\,(1+\alpha)}{(\beta\theta+\delta)\theta^2}\right]\left[\frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta}\right]^2 - 3\left[\frac{[\beta\alpha\theta + \delta(1+\alpha)]}{(\beta\theta+\delta)\theta}\right]^4\right\} \\ &= \frac{(\alpha+1)(\alpha+2)(\alpha+3)(\beta\theta+\delta)^3[\beta\alpha\theta + \delta(\alpha+4)] - 4(\alpha+1)(\alpha+2)(\beta\theta+\delta)^2[(\beta\alpha\theta)^2 + \beta\alpha\theta\delta(\alpha}{\sigma^4\theta^4(\beta\theta+\delta)^4} \\ &\quad \underline{+1) + \beta\alpha\theta\delta(\alpha+3) + \delta^2(\alpha+1)(\alpha+3)] + 6(\alpha+1)(\beta\theta+\delta)[(\beta\alpha\theta)^3 + \beta\alpha\theta\delta^2(\alpha+1)^2 + 2\beta\alpha\theta\delta(\alpha} \\ &\quad \underline{+1) + (\beta\alpha\theta)^2\delta(\alpha+2) + \delta^3(\alpha+1)^2(\alpha+2) + 2\beta\alpha\theta\delta^2(\alpha+1)(\alpha+2)] - 3[(\beta\alpha\theta)^4 + 4(\beta\alpha\theta)^3\delta(\alpha} \\ &\quad \underline{+1) + 6(\beta\alpha\theta\delta)^2(\alpha+1)^2 + 4\beta\alpha\theta\delta^3(\alpha+1)^3 + \delta^4(\alpha+1)^4]}\end{aligned} \tag{3.13}$$

7. The probability function of the k^{th} order statistic for the finite mixture is given by

$$f_k(x) = \frac{n!(\beta\alpha + \delta x)\theta^{\alpha+1}e^{-\theta x}x^{\alpha-1}}{(k-1)!(n-k)!(\beta\theta+\delta)\Gamma(\alpha+1)}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\left[\frac{[\theta\alpha\beta\gamma(\alpha,\theta x) + \delta\gamma(\alpha+1,\theta x)]}{(\beta\theta+\delta)\Gamma(\alpha+1)}\right]^{i+k-1} \tag{3.14}$$

8. Parameter estimation

Method of moments estimation: The parameter estimates of the finite mixture are obtained by solving the below equations simultaneously.

$$\frac{[\beta\alpha\theta+\delta(1+\alpha)]}{(\beta\theta+\delta)\theta} = \bar{x} \tag{3.15}$$

$$\frac{[\beta\alpha\theta+\delta(2+\alpha)]\,(1+\alpha)}{(\beta\theta+\delta)\theta^2} = \frac{\sum_{i=1}^{n} x_i^2}{n} \tag{3.16}$$

$$\frac{[\beta\alpha\theta+\delta(3+\alpha)]\,(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^3} = \frac{\sum_{i=1}^{n} x_i^3}{n} \tag{3.17}$$

$$\frac{[\beta\alpha\theta+\delta(4+\alpha)]\,(3+\alpha)(2+\alpha)(1+\alpha)}{(\beta\theta+\delta)\theta^4} = \frac{\sum_{i=1}^{n} x_i^4}{n} \tag{3.18}$$

Maximum likelihood estimation: The likelihood function of the 4-parameter finite Erlang mixture is given by

$$\begin{aligned} L(\theta,\beta,\delta,\alpha) &= \prod_{i=1}^{n} \frac{\beta\alpha+\delta x_i}{\beta\theta+\delta}\,\frac{\theta^{\alpha+1}e^{-\theta x_i}x_i^{\alpha-1}}{\Gamma(\alpha+1)} \\ &= [(\beta\theta+\delta)\Gamma(\alpha+1)]^{-n}\,\theta^{n(\alpha+1)}e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}[\beta\alpha+\delta x_i] \end{aligned} \tag{3.19}$$

and the log-likelihood function is

$$\text{Ł} = \log L(\theta,\beta,\delta,\alpha) = -n\log[(\beta\theta+\delta)\Gamma(\alpha+1)] + n(\alpha+1)\log\theta - \theta\sum_{i=1}^{n}x_i + (\alpha-1)\sum_{i=1}^{n}\log x_i$$

$$+\sum_{i=1}^{n}\log[\beta\alpha+\delta x_i] \tag{3.20}$$

The partial derivatives with respect to the parameters are as outlined below.

$$\frac{\delta \text{Ł}}{\delta\theta} = -\frac{n\beta}{(\beta\theta+\delta)} + \frac{n(\alpha+1)}{\theta} - \sum_{i=1}^{n}x_i \tag{3.21}$$

$$\frac{\delta \text{Ł}}{\delta\beta} = -\frac{n\theta}{(\beta\theta+\delta)} + \sum_{i=1}^{n}\frac{\alpha}{\beta\alpha+\delta x_i} \tag{3.22}$$

$$\frac{\delta \text{Ł}}{\delta\delta} = -\frac{n}{(\beta\theta+\delta)} + \sum_{i=1}^{n}\frac{x_i}{\beta\alpha+\delta x_i} \tag{3.23}$$

$$\frac{\delta \text{Ł}}{\delta\alpha} = -\frac{n\Gamma'(\alpha+1)}{\Gamma(\alpha+1)} + n\log\theta + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\frac{\beta}{\beta\alpha+\delta x_i} \tag{3.24}$$

The above equations (3.21)-(3.24) are equated to zero and solved simultaneously to obtain estimates of the parameters.

3.2 Special cases

i. when $\delta = 1$, we have Type I 3-parameter Lindley distribution.

$$g(x) = \frac{\theta^{\alpha+1}(\beta\alpha+x)}{\beta\theta}\frac{e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha+1)}, \quad x>0;\theta>0,\beta>0,\alpha=1,2,3,... \tag{3.25}$$

ii. when β=1, we have Type II 3-parameter Lindley distribution.

$$g(x) = \frac{\theta^{\alpha+1}(\alpha+\delta x)}{\theta+\delta}\frac{e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha+1)}, \quad x>0;\theta>0,\delta>0,\alpha=1,2,3,... \tag{3.26}$$

(Zakerzadeh and Dolati[16])

iii. when α=1, we have Type III 3-parameter Lindley distribution.

$$g(x) = \frac{\theta^2(\beta+\delta x)}{\beta\theta+\delta}e^{-\theta x}, \quad x>0;\theta>0,\beta>0,\delta>0 \tag{3.27}$$

iv. when β=δ=1, we have Type I 2-parameter Lindley distribution.

$$g(x) = \frac{\theta^{\alpha+1}(\alpha+x)}{\theta+1}\frac{e^{-\theta x}x^{\alpha-1}}{\Gamma(\alpha+1)}, \quad x>0;\theta>0,,\alpha=1,2,3,... \tag{3.28}$$

v. when α=δ=1, we have Type II 2-parameter Lindley distribution.

$$g(x) = \frac{\theta^2(\beta+x)}{\beta\theta+1}e^{-\theta x}, \quad x>0;\theta>0,\beta>0 \tag{3.29}$$

(Shanker et al.[18])

vi. when α=β=1, we have Type III 2-parameter Lindley distribution.

$$g(x) = \frac{\theta^2(1+\delta x)}{\theta+\delta}e^{-\theta x}, \quad x>0;\theta>0,\delta>0 \tag{3.30}$$

vii. when α=β=δ=1, we have 1-parameter Lindley distribution.

$$g(x) = \frac{\theta^2(1+x)}{\theta+1}e^{-\theta x}, \quad x>0;\theta>0 \tag{3.31}$$

4 The Two Component Four Parameter Finite Erlang Mixture: Case 2

$$\text{Let} \quad g_1(x) = \mathsf{Gamma}(\alpha,\theta), \quad g_2(x) = \mathsf{Gamma}(\alpha+2,\theta), \quad \omega = \frac{\theta^2}{\theta^2+s}, \quad \text{and} \quad (1-\omega) = \frac{s}{\theta^2+s}$$

$$\begin{aligned}
\text{Then} \quad g(x) &= \frac{\theta^2}{\theta^2+s}\frac{\theta^\alpha}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1} + \frac{s}{\theta^2+s}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha+1} = \frac{\theta^{\alpha+2}}{\theta^2+s}e^{-\theta x}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{sx^{\alpha+1}}{\Gamma(\alpha+2)}\right) \\
&= \frac{\theta^{\alpha+2}}{\theta^2+s}e^{-\theta x}\frac{x^{\alpha-1}}{\Gamma(\alpha+2)}[\alpha(\alpha+1)+sx^2] \\
\text{let} \quad s &= \frac{\delta}{\beta} \\
g(x) &= \frac{\theta^{\alpha+2}}{\theta^2+\frac{\delta}{\beta}}e^{-\theta x}\frac{x^{\alpha-1}}{\Gamma(\alpha+2)}\left(\alpha(\alpha+1)+\frac{\delta}{\beta}x^2\right) \\
&= \frac{\beta\alpha(\alpha+1)+\delta x^2}{\beta\theta^2+\delta}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha-1}, \quad x>0;\theta>0,\beta>0,\delta>0,\alpha=1,2,3,...
\end{aligned} \tag{4.1}$$

which is a 4-parameter generalized finite Erlang mixture with the following properties and special cases.

4.1 Properties

1. The CDF of the 4-parameter generalized finite Erlang mixture is given by

$$G(x) = \frac{\theta^2}{\theta^2+s}\frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{s}{\theta^2+s}\frac{\gamma(\alpha+2,\theta x)}{\Gamma(\alpha+2)} = \frac{\left[\theta^2\alpha(\alpha+1)\gamma(\alpha,\theta x) + s\gamma(\alpha+2,\theta x)\right]}{(\theta^2+s)\Gamma(\alpha+2)}$$
$$= \frac{\left[\beta\theta^2\alpha(\alpha+1)\gamma(\alpha,\theta x) + \delta\gamma(\alpha+2,\theta x)\right]}{(\beta\theta^2+\delta)\Gamma(\alpha+2)} \tag{4.2}$$

2. The survival function of the mixed distribution is

$$S(x) = \frac{\theta^2}{\theta^2+s}\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{s}{\theta^2+s}\frac{\Gamma(\alpha+2,\theta x)}{\Gamma(\alpha+2)} = \frac{\left[\theta^2\alpha(\alpha+1)\Gamma(\alpha,\theta x) + s\Gamma(\alpha+2,\theta x)\right]}{(\theta^2+s)\Gamma(\alpha+2)}$$
$$= \frac{\left[\beta\theta^2\alpha(\alpha+1)\Gamma(\alpha,\theta x) + \delta\Gamma(\alpha+2,\theta x)\right]}{(\beta\theta^2+\delta)\Gamma(\alpha+2)} \tag{4.3}$$

3. and the hazard function is

$$h(x) = \frac{\frac{\beta\alpha(\alpha+1)+\delta x^2}{\beta\theta^2+\delta}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha-1}}{\frac{\left[\beta\theta^2\alpha(\alpha+1)\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+2,\theta x)\right]}{(\beta\theta^2+\delta)\Gamma(\alpha+2)}} = \frac{[\beta\alpha(\alpha+1)+\delta x^2]\theta^{\alpha+2}e^{-\theta x}x^{\alpha-1}}{\beta\theta^2\alpha(\alpha+1)\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+2,\theta x)} \tag{4.4}$$

4. The MGF is given by

$$M_x(t) = \frac{\theta^2}{\theta^2+s}\left(\frac{\theta}{\theta-t}\right)^{\alpha} + \frac{s}{\theta^2+s}\left(\frac{\theta}{\theta-t}\right)^{\alpha+2} = \frac{\theta^{\alpha+2}\left[(\theta-t)^{\alpha+2}+s\right]}{(\theta^2+s)(\theta-t)^{\alpha+2}} = \frac{\theta^{\alpha+2}\left[\beta(\theta-t)^{\alpha+2}+\delta\right]}{(\beta\theta^2+\delta)(\theta-t)^{\alpha+2}} \tag{4.5}$$

5. The r^{th} raw moment of the finite mixture is

$$E(X^r) = \frac{\theta^2}{\theta^2+s}\frac{\Gamma(r+\alpha)}{\theta^r\Gamma(\alpha)} + \frac{s}{\theta^2+s}\frac{\Gamma(r+\alpha+2)}{\theta^r\Gamma(\alpha+2)} = \frac{\Gamma(r+\alpha)\left[\theta^2\alpha(\alpha+1)+s(r+\alpha)(r+\alpha+1)\right]}{(\theta^2+s)\theta^r\Gamma(\alpha+2)}$$
$$= \frac{\Gamma(r+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(r+\alpha)(r+\alpha+1)\right]}{(\beta\theta^2+\delta)\theta^r\Gamma(\alpha+2)} \tag{4.6}$$

and the first four raw moments are thus

$$E(X) = \frac{\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta(1+\alpha)} \tag{4.7}$$

$$E(X^2) = \frac{\left[\beta\theta^2\alpha(\alpha+1)+\delta(2+\alpha)(\alpha+3)\right]}{(\beta\theta^2+\delta)\theta^2} \tag{4.8}$$

$$E(X^3) = \frac{(\alpha+2)\left[\beta\theta^2\alpha(\alpha+1)+\delta(3+\alpha)(\alpha+4)\right]}{(\beta\theta^2+\delta)\theta^3} \tag{4.9}$$

$$E(X^4) = \frac{(\alpha+2)(\alpha+3)\left[\beta\theta^2\alpha(\alpha+1)+\delta(4+\alpha)(\alpha+5)\right]}{(\beta\theta^2+\delta)\theta^4} \tag{4.10}$$

6. The central moments are given by

i. Variance

$$\mu_2 = \frac{\Gamma(2+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(2+\alpha)(\alpha+3)\right]}{(\beta\theta^2+\delta)\theta^2\Gamma(\alpha+2)} - \left[\frac{\Gamma(1+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta\Gamma(\alpha+2)}\right]^2$$
$$= \frac{\beta\alpha(\alpha+1)^3\theta^2(\beta\theta^2+\delta)+\delta(\alpha+1)^2(\alpha+2)(\alpha+3)(\beta\theta^2+\delta)-\beta^2\alpha^2(\alpha+1)^2\theta^4-}{\theta^2(\alpha+1)^2(\beta\theta^2+\delta)^2}$$
$$\frac{\delta^2(\alpha+1)^2(\alpha+2)^2-2\beta\alpha(\alpha+1)\theta^2\delta(\alpha+1)(\alpha+2)}{} \tag{4.11}$$

ii. Coefficient of Skewness

$$\frac{\mu_3}{\sigma^3} = \frac{1}{\sigma^3}\left\{\frac{\Gamma(3+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(3+\alpha)(\alpha+4)\right]}{(\beta\theta^2+\delta)\theta^3\Gamma(\alpha+2)} - 3\left[\frac{\Gamma(2+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(2+\alpha)(\alpha+3)\right]}{(\beta\theta^2+\delta)\theta^2\Gamma(\alpha+2)}\right]\right.$$
$$\left.\left[\frac{\Gamma(1+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta\Gamma(\alpha+2)}\right] + 2\left[\frac{\Gamma(1+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta\Gamma(\alpha+2)}\right]^3\right\}$$
$$= \frac{\begin{array}{l}\beta\alpha(\alpha+1)^4(\alpha+2)\theta^2(\beta\theta^2+\delta)^2+\delta(\alpha+1)^3(\alpha+2)(\alpha+3)(\alpha+4)(\beta\theta^2+\delta)^2-3\beta^2\alpha^2(\alpha+1)^4\theta^4 \\ (\beta\theta^2+\delta)-3\beta\alpha(\alpha+1)^4(\alpha+2)\delta\theta^2(\beta\theta^2+\delta)-3\beta\alpha(\alpha+1)^3(\alpha+2)(\alpha+3)\delta\theta^2(\beta\theta^2+\delta)-3(\alpha+1)^3 \\ (\alpha+2)^2(\alpha+3)\delta\theta^2(\beta\theta^2+\delta)+2\beta^3\alpha^3(\alpha+1)^3\theta^6+3\beta^2\alpha^2(\alpha+1)^3(\alpha+2)\theta^4\delta+3\beta\alpha(\alpha+1)^3(\alpha+2)^2 \\ \theta^2\delta^2+\delta^3(\alpha+1)^3(\alpha+2)^3\end{array}}{\sigma^3\theta^3(\alpha+1)^3(\beta\theta^2+\delta)^3} \tag{4.12}$$

iii. Coefficient of Kurtosis

$$\frac{\mu_4}{\sigma^4} = \frac{1}{\sigma^4}\left\{\frac{\Gamma(4+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(4+\alpha)(\alpha+5)\right]}{(\beta\theta^2+\delta)\theta^4\Gamma(\alpha+2)} - 4\left[\frac{\Gamma(3+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(3+\alpha)(\alpha+4)\right]}{(\beta\theta^2+\delta)\theta^3\Gamma(\alpha+2)}\right]\right.$$
$$\left[\frac{\Gamma(1+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta\Gamma(\alpha+2)}\right] + 6\left[\frac{\Gamma(2+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(2+\alpha)(\alpha+3)\right]}{(\beta\theta^2+\delta)\theta^2\Gamma(\alpha+2)}\right]$$
$$\left.\left[\frac{\Gamma(1+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta\Gamma(\alpha+2)}\right]^2 - 3\left[\frac{\Gamma(1+\alpha)\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta\Gamma(\alpha+2)}\right]^4\right\}$$
$$= \frac{\begin{array}{l}\beta\alpha(\alpha+1)^5(\alpha+2)(\alpha+3)\theta^2(\beta\theta^2+\delta)^3+\delta(\alpha+1)^4(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)(\beta\theta^2+\delta)^3-4\beta^2\alpha^2 \\ (\alpha+1)^5(\alpha+2)^2\theta^4(\beta\theta^2+\delta)^2-4\beta\alpha(\alpha+1)^5(\alpha+2)^2\delta\theta^2(\beta\theta^2+\delta)^2-4\beta\alpha(\alpha+1)^4(\alpha+2)(\alpha+3) \\ (\alpha+4)\delta\theta^2(\beta\theta^2+\delta)^2-4(\alpha+1)^4(\alpha+2)^2(\alpha+3)(\alpha+4)\delta^2(\beta\theta^2+\delta)^2+6\beta^3\alpha^3(\alpha+1)^5\theta^6(\beta\theta^2+\delta)+ \\ 6\beta\alpha(\alpha+1)^5(\alpha+2)^2\delta^2\theta^2(\beta\theta^2+\delta)+12\beta^2\alpha^2(\alpha+1)^5(\alpha+2)\delta\theta^4(\beta\theta^2+\delta)+6\beta^2\alpha^2(\alpha+1)^4(\alpha+2) \\ (\alpha+3)\delta\theta^4(\beta\theta^2+\delta)+6(\alpha+1)^4(\alpha+2)^3(\alpha+3)\delta^3(\beta\theta^2+\delta)+12\beta\alpha(\alpha+1)^4(\alpha+2)^2(\alpha+3)\delta^2\theta^2 \\ (\beta\theta^2+\delta)-3\beta^4\alpha^4(\alpha+1)^4\theta^8-12\beta^3\alpha^3(\alpha+1)^4(\alpha+2)\theta^6\delta-18\beta^2\alpha^2(\alpha+1)^4(\alpha+2)^2\theta^4\delta^2- \\ 12\beta\alpha(\alpha+1)^4(\alpha+2)^3\theta^2\delta^3-3\delta^4(\alpha+1)^4(\alpha+2)^4\end{array}}{\sigma^4\theta^4(\alpha+1)^4(\beta\theta^2+\delta)^4} \tag{4.13}$$

7. The probability function of the k^{th} order statistic for the finite mixture is given by

$$f_k(x) = \frac{n![\beta\alpha(\alpha+1)+\delta x^2]\theta^{\alpha+2}e^{-\theta x}x^{\alpha-1}}{(k-1)!(n-k)!(\beta\theta^2+\delta)\Gamma(\alpha+2)}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i\left[\frac{\left[\beta\theta^2\alpha(\alpha+1)\gamma(\alpha,\theta x)+\delta\gamma(\alpha+2,\theta x)\right]}{(\beta\theta^2+\delta)\Gamma(\alpha+2)}\right]^{i+k-1} \tag{4.14}$$

8. Parameter estimation

Method of moments estimation: The method of moments estimators for the parameters of the mixed distribution can be obtained by solving the following equations simultaneously.

$$\frac{\left[\beta\theta^2\alpha(\alpha+1)+\delta(1+\alpha)(\alpha+2)\right]}{(\beta\theta^2+\delta)\theta(1+\alpha)}=\bar{x} \tag{4.15}$$

$$\frac{\left[\beta\theta^2\alpha(\alpha+1)+\delta(2+\alpha)(\alpha+3)\right]}{(\beta\theta^2+\delta)\theta^2}=\frac{1}{n}\sum_{i=1}^{n}x_i^2 \tag{4.16}$$

$$\frac{(\alpha+2)\left[\beta\theta^2\alpha(\alpha+1)+\delta(3+\alpha)(\alpha+4)\right]}{(\beta\theta^2+\delta)\theta^3}=\frac{1}{n}\sum_{i=1}^{n}x_i^3 \tag{4.17}$$

$$\frac{(\alpha+2)(\alpha+3)\left[\beta\theta^2\alpha(\alpha+1)+\delta(4+\alpha)(\alpha+5)\right]}{(\beta\theta^2+\delta)\theta^4}=\frac{1}{n}\sum_{i=1}^{n}x_i^4 \tag{4.18}$$

Maximum likelihood estimation: The likelihood function of the 4-parameter finite Erlang mixture is given by

$$L(\theta,\beta,\delta,\alpha)=\prod_{i=1}^{n}\frac{\beta\alpha(\alpha+1)+\delta x_i^2}{\beta\theta^2+\delta}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x_i}x_i^{\alpha-1}$$
$$=\left[(\beta\theta^2+\delta)\Gamma(\alpha+2)\right]^{-n}\theta^{n(\alpha+2)}e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}\left[\beta\alpha(\alpha+1)+\delta x_i^2\right] \tag{4.19}$$

and the log-likelihood function is

$$Ł=\log L(\theta,\beta,\delta,\alpha)=-n\log\left[(\beta\theta^2+\delta)\Gamma(\alpha+2)\right]+n(\alpha+2)\log\theta-\theta\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\log x_i+$$
$$\sum_{i=1}^{n}\log\left[\beta\alpha(\alpha+1)+\delta x_i^2\right] \tag{4.20}$$

The partial derivatives with respect to the parameters are obtained and the resulting equations are as follows;

$$\frac{\delta Ł}{\delta\theta}=-\frac{2n\beta\theta}{(\beta\theta^2+\delta)}+\frac{n(\alpha+2)}{\theta}-\sum_{i=1}^{n}x_i \tag{4.21}$$

$$\frac{\delta Ł}{\delta\beta}=-\frac{n\theta^2}{(\beta\theta^2+\delta)}+\sum_{i=1}^{n}\frac{\alpha(\alpha+1)}{\beta\alpha(\alpha+1)+\delta x_i^2} \tag{4.22}$$

$$\frac{\delta Ł}{\delta\delta}=-\frac{n}{(\beta\theta^2+\delta)}+\sum_{i=1}^{n}\frac{x_i^2}{\beta\alpha(\alpha+1)+\delta x_i^2} \tag{4.23}$$

$$\frac{\delta Ł}{\delta\alpha}=-\frac{n\Gamma'(\alpha+2)}{\Gamma(\alpha+2)}+n\log\theta+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\frac{\beta(\alpha+1)+\beta\alpha}{\beta\alpha(\alpha+1)+\delta x_i^2} \tag{4.24}$$

The maximum likelihood estimates of the parameters are obtained by equating equations (4.21)-(4.24) to zero and solving them simultaneously.

4.2 Special cases

i. when $\delta=1$, we have type I 3-parameter finite Erlang mixture.

$$g(x)=\frac{\beta\alpha(\alpha+1)+x^2}{\beta\theta^2+1}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha-1},\quad x>0;\theta>0,\beta>0,\alpha=1,2,3,... \tag{4.25}$$

ii. when $\beta = 1$, we have type II 3-parameter finite Erlang mixture.

$$g(x) = \frac{\alpha(\alpha+1)+\delta x^2}{\theta^2+\delta}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha-1}, \quad x>0; \theta>0, \delta>0, \alpha=1,2,3,... \tag{4.26}$$

iii. when $\alpha = 1$, we have type III 3-parameter finite Erlang mixture.

$$g(x) = \frac{2\beta+\delta x^2}{\beta\theta^2+\delta}\frac{\theta^3}{2}e^{-\theta x}, \quad x>0; \theta>0, \beta>0, \delta>0 \tag{4.27}$$

iv. when $\beta = \delta = 1$, we have type I 2-parameter finite Erlang mixture.

$$g(x) = \frac{\alpha(\alpha+1)+x^2}{\theta^2+1}\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha-1}, \quad x>0; \theta>0, \alpha=1,2,3,... \tag{4.28}$$

v. when $\alpha = \delta = 1$, we have type II 2-parameter finite Erlang mixture.

$$g(x) = \frac{2\beta+x^2}{\beta\theta^2+1}\frac{\theta^3}{2}e^{-\theta x}, \quad x>0; \theta>0, \beta>0 \tag{4.29}$$

vi. when $\alpha = \beta = 1$, we have type III 2-parameter finite Erlang mixture.

$$g(x) = \frac{2+\delta x^2}{\theta^2+\delta}\frac{\theta^3}{2}e^{-\theta x}, \quad x>0; \theta>0, \delta>0 \tag{4.30}$$

When $\delta = 2$, the type III 2-parameter finite Erlang mixture becomes the Akash distribution. (Shanker[26]).

vii. when $\alpha = \beta = \delta = 1$, we have 1-parameter finite Erlang mixture.

$$g(x) = \frac{2+x^2}{\theta^2+1}\frac{\theta^3}{2}e^{-\theta x}, \quad x>0; \theta>0 \tag{4.31}$$

5 The Two Component Four Parameter Finite Erlang Mixture: Case 3

Let $g_1(x) = \text{Gamma}(\alpha,\theta)$, $g_2(x) = \text{Gamma}(\alpha+3,\theta)$, $\omega = \frac{\theta^3}{\theta^3+s}$, and $(1-\omega) = \frac{s}{\theta^3+s}$

$$\text{Then} \quad g(x) = \frac{\theta^3}{\theta^3+s}\frac{\theta^\alpha}{\Gamma(\alpha)}e^{-\theta x}x^{\alpha-1} + \frac{s}{\theta^3+s}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x}x^{\alpha+2} = \frac{\theta^{\alpha+3}}{\theta^3+s}e^{-\theta x}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \frac{sx^{\alpha+2}}{\Gamma(\alpha+3)}\right)$$

$$= \frac{\theta^{\alpha+3}}{\theta^3+s}e^{-\theta x}\frac{x^{\alpha-1}}{\Gamma(\alpha+3)}[\alpha(\alpha+1)(\alpha+2)+sx^3]$$

Let $s = \frac{\delta}{\beta}$

$$g(\lambda) = \frac{\theta^{\alpha+3}}{\theta^3+\frac{\delta}{\beta}}e^{-\theta x}\frac{x^{\alpha-1}}{\Gamma(\alpha+3)}\left(\alpha(\alpha+1)(\alpha+2)+\frac{\delta}{\beta}x^3\right)$$

$$= \frac{\beta\alpha(\alpha+1)(\alpha+2)+\delta x^3}{\beta\theta^3+\delta}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x}x^{\alpha-1}, \quad x>0; \theta>0, \beta>0, \delta>0, \alpha=1,2,3,... \tag{5.1}$$

which is a 4-parameter generalized finite Erlang mixture with the following properties and special cases.

5.1 Properties

1. The CDF of the 4-parameter generalized finite Erlang mixture is

$$G(x) = \frac{\theta^3}{\theta^3+s}\frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{s}{\theta^3+s}\frac{\gamma(\alpha+3,\theta x)}{\Gamma(\alpha+3)} = \frac{\theta^3\alpha(\alpha+1)(\alpha+2)\gamma(\alpha,\theta x)+s\gamma(\alpha+3,\theta x)}{(\theta^3+s)\Gamma(\alpha+3)}$$
$$= \frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)\gamma(\alpha,\theta x)+\delta\gamma(\alpha+3,\theta x)}{(\beta\theta^3+\delta)\Gamma(\alpha+3)} \tag{5.2}$$

2. and the survival function is

$$S(x) = \frac{\theta^3}{\theta^3+s}\frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{s}{\theta^3+s}\frac{\Gamma(\alpha+3,\theta x)}{\Gamma(\alpha+3)} = \frac{\theta^3\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha,\theta x)+s\Gamma(\alpha+3,\theta x)}{(\theta^3+s)\Gamma(\alpha+3)}$$
$$= \frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+3,\theta x)}{(\beta\theta^3+\delta)\Gamma(\alpha+3)} \tag{5.3}$$

3. The hazard function is given by

$$h(x) = \frac{\frac{\beta\alpha(\alpha+1)(\alpha+2)+\delta x^3}{\beta\theta^3+\delta}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x}x^{\alpha-1}}{\frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+3,\theta x)}{(\beta\theta^3+\delta)\Gamma(\alpha+3)}} = \frac{[\beta\alpha(\alpha+1)(\alpha+2)+\delta x^3]\theta^{\alpha+3}e^{-\theta x}x^{\alpha-1}}{\beta\theta^3\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha,\theta x)+\delta\Gamma(\alpha+3,\theta x)} \tag{5.4}$$

4. and the MGF is

$$M_x(t) = \frac{\theta^3}{\theta^3+s}\left(\frac{\theta}{\theta-t}\right)^{\alpha} + \frac{s}{\theta^3+s}\left(\frac{\theta}{\theta-t}\right)^{\alpha+3} = \frac{\theta^{\alpha+3}[(\theta-t)^3+s]}{(\theta^3+s)(\theta-t)^{\alpha+3}} = \frac{\theta^{\alpha+3}[\beta(\theta-t)^3+\delta]}{(\beta\theta^3+\delta)(\theta-t)^{\alpha+3}} \tag{5.5}$$

5. The r^{th} raw moment of the 4-parameter generalized finite Erlang mixture is

$$E(X^r) = \frac{\theta^3}{\theta^3+s}\frac{\Gamma(r+\alpha)}{\theta^r\Gamma(\alpha)} + \frac{s}{\theta^3+s}\frac{\Gamma(r+\alpha+3)}{\theta^r\Gamma(\alpha+3)}$$
$$= \frac{\Gamma(r+\alpha)[\theta^3\alpha(\alpha+1)(\alpha+2)+s(r+\alpha)(r+\alpha+1)(r+\alpha+2)]}{(\theta^3+s)\theta^r\Gamma(\alpha+3)}$$
$$= \frac{\Gamma(r+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(r+\alpha)(r+\alpha+1)(r+\alpha+2)]}{(\beta\theta^3+\delta)\theta^r\Gamma(\alpha+3)} \tag{5.6}$$

and in particular the first four raw moments are

$$E(X) = \frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)}{(\beta\theta^3+\delta)\theta(\alpha+2)(\alpha+1)} \tag{5.7}$$

$$E(X^2) = \frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(2+\alpha)(\alpha+3)(\alpha+4)}{(\beta\theta^3+\delta)\theta^2(\alpha+2)} \tag{5.8}$$

$$E(X^3) = \frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(3+\alpha)(\alpha+4)(\alpha+5)}{(\beta\theta^3+\delta)\theta^3} \tag{5.9}$$

$$E(X^4) = \frac{(3+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(4+\alpha)(\alpha+5)(\alpha+6)]}{(\beta\theta^3+\delta)\theta^4} \tag{5.10}$$

6. The central moments are given by

    i. Variance

$$\mu_2 = \frac{\Gamma(2+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(2+\alpha)(\alpha+3)(\alpha+4)]}{(\beta\theta^3+\delta)\theta^2\Gamma(\alpha+3)} - \left[\frac{\Gamma(1+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)]}{(\beta\theta^3+\delta)\theta\Gamma(\alpha+3)}\right]^2$$

$$= \frac{\beta\alpha(\alpha+1)^3(\alpha+2)^2\theta^3(\beta\theta^3+\delta)+\delta(\alpha+1)^2(\alpha+2)^2(\alpha+3)(\alpha+4)(\beta\theta^3+\delta) - \beta^2\alpha^2(\alpha+1)^2(\alpha+2)^2\theta^6-\delta^2(\alpha+1)^2(\alpha+2)^2(\alpha+3)^2-2\beta\delta\alpha(\alpha+1)^2(\alpha+2)^2(\alpha+3)\theta^3}{\theta^2(\alpha+1)^2(\alpha+2)^2(\beta\theta^3+\delta)^2} \tag{5.11}$$

    ii. Coefficient of Skewness

$$\frac{\mu_3}{\sigma^3} = \frac{1}{\sigma^3}\Bigg\{\frac{\Gamma(3+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(3+\alpha)(\alpha+4)(\alpha+5)]}{(\beta\theta^3+\delta)\theta^3\Gamma(\alpha+3)} - 3\left[\frac{\Gamma(2+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(2+\alpha)(\alpha+3)(\alpha+4)]}{(\beta\theta^3+\delta)\theta^2\Gamma(\alpha+3)}\right]\left[\frac{\Gamma(1+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)]}{(\beta\theta^3+\delta)\theta\Gamma(\alpha+3)}\right] + 2\left[\frac{\Gamma(1+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)]}{(\beta\theta^3+\delta)\theta\Gamma(\alpha+3)}\right]^3\Bigg\}$$

$$= \frac{\begin{aligned}&\beta\alpha(\alpha+1)^4(\alpha+2)^4\theta^3(\beta\theta^3+\delta)^2+\delta(\alpha+1)^3(\alpha+2)^3(\alpha+3)(\alpha+4)(\alpha+5)(\beta\theta^3+\delta)^2-\\&3\beta^2\alpha^2(\alpha+1)^4(\alpha+2)^3\theta^6(\beta\theta^3+\delta)-3\beta\delta\alpha(\alpha+1)^2(\alpha+2)^2(\alpha+3)\theta^3(\beta\theta^3+\delta)-3\beta\delta\alpha(\alpha+1)^3\\&(\alpha+2)^3(\alpha+3)(\alpha+4)\theta^3(\beta\theta^3+\delta)-3\delta^2(\alpha+1)^3(\alpha+2)^3(\alpha+3)^2(\alpha+4)(\beta\theta^3+\delta)+2\beta^3\alpha^3\\&(\alpha+1)^3(\alpha+2)^3\theta^6+6\beta^2\delta\alpha^2(\alpha+1)^3(\alpha+2)^3(\alpha+3)\theta^6+6\beta\delta^2\alpha(\alpha+1)^3(\alpha+2)^2(\alpha+3)^2\theta^3\delta^3\\&(\alpha+1)^3(\alpha+2)^3(\alpha+3)^3\end{aligned}}{\sigma^3\theta^3(\alpha+1)^3(\alpha+2)^3(\beta\theta^3+\delta)^3} \tag{5.12}$$

    iii. Coefficient of Kurtosis

$$\frac{\mu_4}{\sigma^4} = \frac{1}{\sigma^4}\Bigg\{\frac{\Gamma(4+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(4+\alpha)(\alpha+5)(\alpha+6)]}{(\beta\theta^3+\delta)\theta^4\Gamma(\alpha+3)} - 4\left[\frac{\Gamma(3+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(3+\alpha)(\alpha+4)(\alpha+5)]}{(\beta\theta^3+\delta)\theta^3\Gamma(\alpha+3)}\right]\left[\frac{\Gamma(1+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)]}{(\beta\theta^3+\delta)\theta\Gamma(\alpha+3)}\right] + 6\left[\frac{\Gamma(2+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(2+\alpha)(\alpha+3)(\alpha+4)]}{(\beta\theta^3+\delta)\theta^2\Gamma(\alpha+3)}\right]\left[\frac{\Gamma(1+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)]}{(\beta\theta^3+\delta)\theta\Gamma(\alpha+3)}\right]^2 - 3\left[\frac{\Gamma(1+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)]}{(\beta\theta^3+\delta)\theta\Gamma(\alpha+3)}\right]^4\Bigg\}$$

$$= \frac{\begin{aligned}&\beta\alpha(\alpha+1)^5(\alpha+2)^5(\alpha+3)\theta^3(\beta\theta^3+\delta)^3+\delta(\alpha+1)^4(\alpha+2)^4(\alpha+3)(\alpha+4)(\alpha+5)(\alpha+6)\\&(\beta\theta^3+\delta)^3-4\beta^2\alpha^2(\alpha+1)^5(\alpha+2)^5\theta^6(\beta\theta^3+\delta)^2+4\beta\delta\alpha(\alpha+1)^5(\alpha+2)^5(\alpha+3)\theta^3(\beta\theta^3+\delta)^2+\\&4\beta\delta\alpha(\alpha+1)^4(\alpha+2)^4(\alpha+3)(\alpha+4)(\alpha+5)\theta^3(\beta\theta^3+\delta)^2+4\delta^2(\alpha+1)^4(\alpha+2)^4(\alpha+3)^2(\alpha+4)\\&(\alpha+5)(\beta\theta^3+\delta)^2+6\beta^3\alpha^3(\alpha+1)^5(\alpha+2)^4\theta^9(\beta\theta^3+\delta)+6\beta\delta^2\alpha(\alpha+1)^5(\alpha+2)^4(\alpha+3)^2\theta^3\\&(\beta\theta^3+\delta)+12\beta^2\delta\alpha^2(\alpha+1)^5(\alpha+2)^4(\alpha+3)\theta^6(\beta\theta^3+\delta)+6\beta^2\delta\alpha^2(\alpha+1)^4(\alpha+2)^4(\alpha+3)(\alpha+4)\\&\theta^6(\beta\theta^3+\delta)+6\delta^3(\alpha+1)^4(\alpha+2)^4(\alpha+3)^3(\alpha+4)(\beta\theta^3+\delta)+12\beta\delta^2\alpha(\alpha+1)^4(\alpha+2)^4(\alpha+3)^2\\&(\alpha+4)\theta^3(\beta\theta^3+\delta)-3\beta^4\alpha^4(\alpha+1)^4(\alpha+2)^4\theta^{12}+12\beta^3\delta\alpha^3(\alpha+1)^4(\alpha+2)^4(\alpha+3)\theta^9+18\beta^2\delta^2\alpha^2\\&(\alpha+1)^4(\alpha+2)^4(\alpha+3)^2\theta^6+12\beta\delta^3\alpha(\alpha+1)^4(\alpha+2)^4(\alpha+3)^3\theta^3+3\delta^4(\alpha+1)^4(\alpha+2)^4(\alpha+3)^4\end{aligned}}{\theta^4(\alpha+1)^4(\alpha+2)^4(\beta\theta^3+\delta)^4} \tag{5.13}$$

7. The probability function of the k^{th} order statistic for the finite mixture is given by

$$f_k(x) = \frac{n![\beta\alpha(\alpha+1)(\alpha+2)+\delta x^3]\theta^{\alpha+3}e^{-\theta x}x^{\alpha-1}}{(k-1)!(n-k)![\beta\theta^3+\delta]\Gamma(\alpha+3)}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^i$$
$$\left[\frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)\gamma(\alpha,\theta x)+\delta\gamma(\alpha+3,\theta x)}{(\beta\theta^3+\delta)\Gamma(\alpha+3)}\right]^{i+k-1} \quad (5.14)$$

8. Parameter estimation

Method of moments estimation: By solving the equations below simultaneously, the estimators of the mixture distribution are obtained.

$$\frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(1+\alpha)(\alpha+2)(\alpha+3)}{(\beta\theta^3+\delta)\theta(\alpha+1)(\alpha+2)} = \bar{x} \quad (5.15)$$

$$\frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(2+\alpha)(\alpha+3)(\alpha+4)}{(\beta\theta^3+\delta)\theta^2(\alpha+2)} = \frac{1}{n}\sum_{i=1}^{n}x_i^2 \quad (5.16)$$

$$\frac{\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(3+\alpha)(\alpha+4)(\alpha+5)}{(\beta\theta^3+\delta)\theta^3} = \frac{1}{n}\sum_{i=1}^{n}x_i^3 \quad (5.17)$$

$$\frac{(3+\alpha)[\beta\theta^3\alpha(\alpha+1)(\alpha+2)+\delta(4+\alpha)(\alpha+5)(\alpha+6)]}{(\beta\theta^3+\delta)\theta^4} = \frac{1}{n}\sum_{i=1}^{n}x_i^4 \quad (5.18)$$

Maximum likelihood estimation
The likelihood function of the 4-parameter finite Erlang mixture is

$$L(\theta,\beta,\delta,\alpha) = \prod_{i=1}^{n}\frac{\beta\alpha(\alpha+1)(\alpha+2)+\delta x_i^3}{\beta\theta^3+\delta}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x_i}x_i^{\alpha-1}$$
$$= \left[(\beta\theta^3+\delta)\Gamma(\alpha+3)\right]^{-n}\theta^{n(\alpha+3)}e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}x_i^{\alpha-1}\prod_{i=1}^{n}\left[\beta\alpha(\alpha+1)(\alpha+2)+\delta x_i^3\right] \quad (5.19)$$

and the log-likelihood function is

$$Ł = \mathsf{log}L(\theta,\beta,\delta,\alpha) = -n\mathsf{log}\left[(\beta\theta^3+\delta)\Gamma(\alpha+3)\right]+n(\alpha+3)\mathsf{log}\theta-\theta\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\mathsf{log}x_i+$$
$$\sum_{i=1}^{n}\mathsf{log}\left[\beta\alpha(\alpha+1)(\alpha+2)+\delta x_i^3\right] \quad (5.20)$$

The derivatives with respect to respective parameters are

$$\frac{\delta Ł}{\delta\theta} = -\frac{3n\beta\theta^2}{(\beta\theta^3+\delta)}+\frac{n(\alpha+3)}{\theta}-\sum_{i=1}^{n}x_i \quad (5.21)$$

$$\frac{\delta Ł}{\delta\beta} = -\frac{n\theta^3}{(\beta\theta^3+\delta)}+\sum_{i=1}^{n}\frac{\alpha(\alpha+1)(\alpha+2)}{\beta\alpha(\alpha+1)(\alpha+2)+\delta x_i^3} \quad (5.22)$$

$$\frac{\delta Ł}{\delta\delta} = -\frac{n}{(\beta\theta^3+\delta)}+\sum_{i=1}^{n}\frac{x_i^3}{\beta\alpha(\alpha+1)(\alpha+2)+\delta x_i^3} \quad (5.23)$$

$$\frac{\delta Ł}{\delta\alpha} = -\frac{n\Gamma'(\alpha+3)}{\Gamma(\alpha+3)}+n\mathsf{log}\theta+\sum_{i=1}^{n}\mathsf{log}x_i+\sum_{i=1}^{n}\frac{\beta(\alpha+1)(\alpha+2)+\beta\alpha(\alpha+2)+\beta\alpha(\alpha+1)}{\beta\alpha(\alpha+1)(\alpha+2)+\delta x_i^3} \quad (5.24)$$

By equating the above equations (5.21)-(5.24) to zero and solving them simultaneously, the maximum likelihood estimates of the parameters are obtained.

5.2 Special cases

i. when $\delta = 1$, we have type I 3-parameter finite Erlang mixture.

$$g(x) = \frac{\beta\alpha(\alpha+1)(\alpha+2)+x^3}{\beta\theta^3+1}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x}x^{\alpha-1}, \quad x>0;\theta>0,\beta>0,\alpha=1,2,3,... \tag{5.25}$$

ii. when $\beta = 1$, we have type II 3-parameter finite Erlang mixture.

$$g(x) = \frac{\alpha(\alpha+1)(\alpha+2)+\delta x^3}{\theta^3+\delta}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x}x^{\alpha-1}, \quad x>0;\theta>0,\delta>0,\alpha=1,2,3,... \tag{5.26}$$

iii. when $\alpha = 1$, we have type III 3-parameter finite Erlang mixture.

$$g(x) = \frac{6\beta+\delta x^3}{\beta\theta^3+\delta}\frac{\theta^4}{6}e^{-\theta x}, \quad x>0;\theta>0,\beta>0,\delta>0 \tag{5.27}$$

iv. when $\beta = \delta = 1$, we have type I 2-parameter finite Erlang mixture.

$$g(x) = \frac{\alpha(\alpha+1)(\alpha+2)+x^3}{\theta^3+1}\frac{\theta^{\alpha+3}}{\Gamma(\alpha+3)}e^{-\theta x}x^{\alpha-1}, \quad x>0;\theta>0,\alpha=1,2,3,... \tag{5.28}$$

v. when $\alpha = \delta = 1$, we have type II 2-parameter finite Erlang mixture.

$$g(x) = \frac{6\beta+x^3}{\beta\theta^3+1}\frac{\theta^4}{6}e^{-\theta x}, \quad x>0;\theta>0,\beta>0 \tag{5.29}$$

vi. when $\alpha = \beta = 1$, we have type III 2-parameter finite Erlang mixture.

$$g(x) = \frac{6+\delta x^3}{\theta^3+\delta}\frac{\theta^4}{6}e^{-\theta x}, \quad x>0;\theta>0,\delta>0 \tag{5.30}$$

When $\delta = 6$, the distribution is a one parameter Rama. (Shanker[28])

vii. when $\alpha = \beta = \delta = 1$, we have 1-parameter finite Erlang mixture.

$$g(x) = \frac{6+x^3}{\theta^3+1}\frac{\theta^4}{6}e^{-\theta x}, \quad x>0;\theta>0 \tag{5.31}$$

6 Application

An application of the three cases of the generalized finite Erlang mixture has been demonstrated in this section. To assess and compare their goodness of fit, the three mixed distributions have been fitted to data on the relief times in minutes of patients receiving an analgesic. The data set was provided by Gross and Clark[30] in 1975 and has since been applied by various authors including Shanker[26] in studying the Shanker distribution, Shanker[11] in fitting the Amarendra distribution, Nwikpe and Iwok[10] in evaluating a three-parameter Sujatha distribution, and Oguntunde[31] in assessing the generalized inverse exponential distribution. The -2 log-likelihood (-2ln(L)), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Akaike Information Criterion Corrected (AICC) statistics have been used for comparison, and they are computed using the formulae below.

$AIC = -2lnL + 2k$, $BIC = -2lnL + klnn$, $AICC = AIC + \frac{2k(k+1)}{n-k-1}$, where k is the number of parameters and n is the sample size.

Data set: The relief times (in minutes) of 20 patients receiving an analgesic
1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3.0,1.7,2.3,1.6,2.0

The finite Erlang mixtures give better fits compared to the rest of the finite mixtures, as seen in the statistics used in the comparison, which are lower for the mixed finite Erlang distributions compared to those of the Akash, exponential, Shanker, Lindley, Sujatha, and Amarendra distributions. Case 2 gives a better fit than case 1, while case 3 gives the best fit among the 3 cases of the mixed Erlang distributions.

Table 1. Parameters estimates, -2ln(L), AIC, BIC and AICC statistics for the 3 cases of the 2-component 4-parameter Erlang mixture, and other mixed finite distributions using the relief times' data set

Distribution		Estimated parameters				-2ln(L)	AIC	BIC	AICC
Akash distribution		-	$\hat{\theta}$=1.1569	-	-	59.52	61.52	62.52	61.74
Exponential distribution		-	$\hat{\theta}$=0.5263	-	-	65.67	67.67	68.67	67.9
Shanker distribution		-	$\hat{\theta}$=0.8039	-	-	59.78	61.78	62.78	62.01
Lindley distribution		-	$\hat{\theta}$=0.8161	-	-	60.5	62.5	63.5	62.72
Sujatha distribution		-	$\hat{\theta}$=1.1367	-	-	57.5	59.5	60.5	59.72
Amarendra distribution		-	$\hat{\theta}$=1.4808	-	-	55.64	57.64	58.63	57.86
2-component	Case 1	$\hat{\alpha}$=8	$\hat{\theta}$=5.1	$\hat{\beta}$=0.001	$\hat{\delta}$=2.87	36.7	44.7	48.68	47.37
4-parameter	Case 2	$\hat{\alpha}$=7	$\hat{\theta}$=5.08	$\hat{\beta}$=0.0001	$\hat{\delta}$=3.69	36.6	44.6	48.58	47.27
Erlang mixture	Case 3	$\hat{\alpha}$=10	$\hat{\theta}$=5.52	$\hat{\beta}$=0.15	$\hat{\delta}$=3.42	35.45	43.45	47.43	46.12

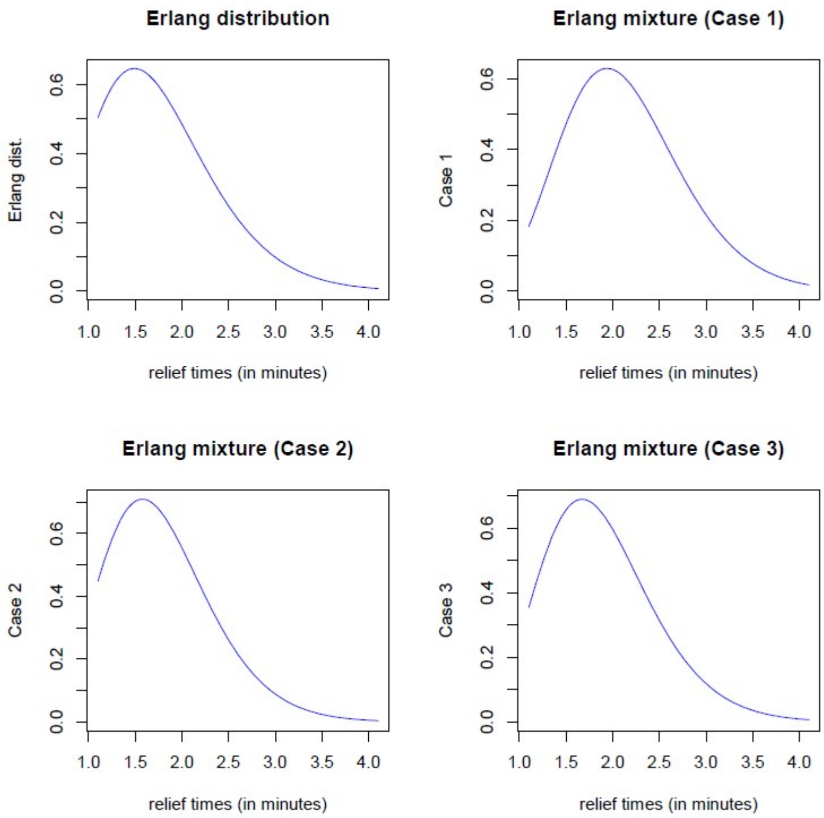

Fig. 1. The pdfs for the Erlang distribution and the 3 cases of the mixed Erlang finite mixture using the relief times' data set.

7 Conclusion

Three cases of the two-component 4-parameter generalized Erlang mixture have been derived, each with varying mixing weights and component probability (Erlang) distributions. These include their special cases: the two-component 1, 2, and 3-parameter finite Erlang mixtures. The properties of these mixtures, including the distribution function, survival function, hazard function, moment generating function, raw and central moments, mean, variance, coefficient of skewness, coefficient of kurtosis, and order statistics, have been thoroughly explored. Parameter estimation was conducted using the method of moments and maximum likelihood estimation.

Applying the finite Erlang mixtures to a real dataset demonstrated that they provide superior fits compared to other mixed finite distributions.

For future research, it is recommended to construct additional finite Erlang mixtures using various mixing weights and probability distributions, and to further explore their applications in different contexts.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Pearson K. Contributions to the mathematical theory of evolution. Philosophical Transactions of the Royal Society of London. 1894;71-110.

[2] Lindley D. Fiducial distributions and Bayes' theorem. Journal of the Royal Statistical Society. Series B (Methodological). 1958;102-107.

[3] Shanker R. Others Sujatha distribution and its applications. Statistics in Transition. New Series. 2016;17:391-410.

[4] Shanker R. A quasi Sujatha distribution. International Journal of Probability and Statistics. 2016;5:9-100.

[5] Shanker R, Shukla K, Fesshaye H. A generalization of Sujatha distribution and its applications with real lifetime data. Journal of Institute of Science and Technology. 2017;22:66-83.

[6] Shanker R, Shukla K, Shanker, R. A quasi Aradhana distribution with properties and application. International Journal of Statistics and Systems. 2018;13:61-80.

[7] Mussie T, Shanker R. A two-parameter Sujatha distribution. Biometrics and Biostatistics International Journal. 2018;7:188-197.

[8] Tesfay M, Shanker R. A new-two parameter Sujatha distribution with properties and applications. Türkiye Klinikleri Biyoistatistik. 2018;10:96-113.

[9] Tesfay M, Shanker, R. Another Two-Parameter Sujatha distribution with properties and applications. Journal of Mathematical Sciences and Modelling. 2019;2:1-13.

[10] Nwikpe B, Iwok I. A Three-parameter Sujatha distribution with application. European-American Journal. 2020;8:22-34.

[11] Shanker R. Amarendra distribution and its applications. American Journal of Mathematics and Statistics. 2016;6:44-56.

[12] Shanker R. Akshaya distribution and its applications. American Journal of Mathematics and Statistics. 2017;7:1-59

[13] Rashid Q, Yaseen D, Jamshaid M. Quasi Amarendra distribution and Its Properties. 2019;7:490-502.

[14] Shanker R. Devya distribution and its applications. International Journal of Statistics and Applications. 2016;6:189-202.

[15] Shanker R. Shambhu distribution and its applications. International Journal of Probability and Statistics. 2016;5:48-63.

[16] Zakerzadeh H, Dolati A. Generalized Lindley distribution. Journal of Mathematical Extension; 2009.

[17] Nadarajah S, Bakouch H, Tahmasbi R. A generalized Lindley distribution. Sankhya B. 2011;73:331-359.

[18] Shanker R, Sharma S, Shanker R. A two-parameter Lindley distribution for modeling waiting and survival times data. Scientific Research Publishing; 2013.

[19] Shanker R, Mishra A. A two-parameter Lindley distribution. Statistics In Transition New Series. 2013;14:45-56.

[20] Shanker R, Mishra A. A quasi Lindley distribution. African Journal of Mathematics and Computer Science Research. 2013;6:64-71.

[21] Shanker R, Amanuel A. A new quasi Lindley distribution. International Journal of Statistics and Systems. 2013;8:143-156.

[22] Merovci F. Transmuted Lindley distribution. Int. J. Open Problems Compt. Math. 2013;6:63-72.

[23] Elbatal I, Merovi F, Elgarhy M. A new generalized Lindley distribution. Mathematical Theory and Modeling. 2013;3:30-47.

[24] Oluyede B, Yang T. A new class of generalized Lindley distributions with applications. Journal of Statistical Computation and Simulation. 2015;85:2072-2100.

[25] Abouammoh A, Alshangiti A, Ragab I. A new generalized Lindley distribution. Journal of Statistical Computation and Simulation. 2015;85:3662-3678.

[26] Shanker R. Shanker distribution and its applications. International Journal of Statistics and Applications. 2015;5:338-348.

[27] Shanker R. Akash distribution and its appplications. International Journal of Probability and Statistics. 2015;4:65-75.

[28] Shanker, R. Rama distribution and its applications. International Journal of Statistics and Applications. 2017;7:26-35.

[29] Shanker R. Suja distribution and its application. International Journal of Probability and Statistics. 2017;6:11-19.

[30] Gross A, Clark V. Survival distributions: Reliability applications in the biomedical sciences; 1975.

[31] Oguntunde P, Adejumo A. The generalized inverted generalized exponential distribution with an application to a censored data. Journal of Statistics Applications and Probability. 2015;4:223.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/122335