Al Azhar Bulletin of Science Vol. (25) No.1, June, 7 - 14, 2014,.

STUDYING STABILITY OF THE LIBERATION POINTS OFBINARY ASTEROIDES

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ABSTRACT

In this study, the locations of the equilibrium points of both triangular and collinear of restricted three-body problem and their stabilities are studied.This study was applied on ten binary asteroids. A code was constructed by MATHEMATICA language to obtain liberation points and their stabilities.

On the other hand, the contour of zero velocity curves was displayed for two stable and unstable binaries.

Key words: binary asteroids, liberation points, stability.

1. INTRODUCTION

Euler(1773)has discovered the three co-linear Lagrangian points (L_1,L_2 and L_3) ,a few years later Lagrange discovered the remaining points (L_4 and L_5). Sharma(1980) Studied periodic orbits of the second kind in the restricted three body problem when the more massive primary is an oblate spheroid.Sharivastavaet al.,(1983) studied equation of motion of the restricted problem of three bodies with variable mass.Gabernet al.,(1991)studied a restricted four body model for the dynamic near the lagrangian points of the sun-Jupiter system.Mathlouthi(1998) Studied the infinity of periodic solution of the restricted three body problem by using a variational formulation.Llibre(1999) studied periodic and qusi-periodic orbits of the spatial three body problem. Llibreet al.,(2003) studied periodic orbits of the planer collision restricted three body problem. Llibreet al.,(2003) studied,symmetric periodic orbits of a collinear restricted three body problem. Munzoet al.,(2004) studied the families of symmetric periodic orbits in the three body problem and figure eight.Inga Jinanget al.,(2004) studied modified restricted three body problem.Reppert (2006)investigated how to refine patched conic approximation to the restricted four body problem.

In this study, the locations of the collinear and triangular points of ten binary asteroids have been computed and their stabilitieshave been determined.

2. Equation Of Motion Of TheRestricted Three-Body Problem

The restricted three-body problem refers to the dynamics of two bodies of masses m1 ≤ m2 (referred to as the primaries) that move along circles about their common center of mass, and of a third body, of infinitesimal mass, that is subject to the gravitational attraction of the primaries. The motion of the primaries is not affected by the motion of the infinitesimal mass. Fig.(2.1) illustrates the position of the third body $m_3$referring to the center of mass of m_1 and m_2, and the reference plane (x, y, z).

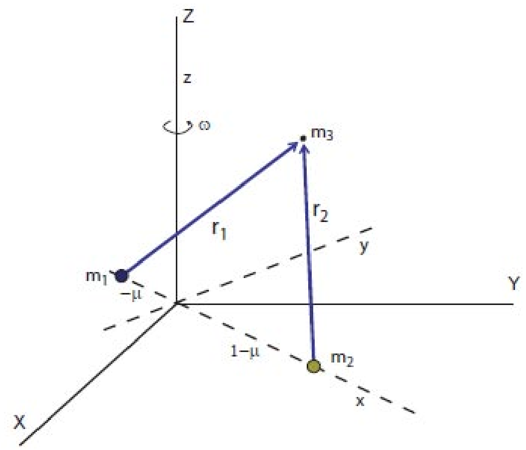

Figure2.1 inertial frame of three bodies.

The equations of motion for third body in synodic barycentric coordinate are given by (Szebehely, 1967):

$$\ddot{x} - 2\dot{y} = x - \frac{\mu_1}{r_1^3}(x - x_1) - \frac{\mu_2}{r_2^3}(x - x_2) \ , \tag{2.1}$$

$$\ddot{y} + 2\dot{x} = y - \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right]y, \qquad (2.2)$$

$$\ddot{z} = -\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] z \quad . \qquad (2.3)$$

where

$\mu = \frac{m_2}{m_1+m_2}$: mass ratio

$r_1^2 = (x - x_1)^2 + y^2 + z^2$ distance from m_1 to m_3,

$r_2^2 = (x - x_2)^2 + y^2 + z^2$:distance from m_2 to m_3,

$\mu_1 = Gm_1 = 1 - \mu$: gravitational parameter for m_1,

$\mu_2 = Gm_2 = \mu$:gravitational parameter for m_2,

$x_1 = -\mu_2 = -\mu$:distance from m_1 to mass center,

$x_2 = \mu_1 = 1 - \mu$: distance from m_2 to mass center.

3. Liberation points

At the liberation points there are zero velocity regions. So that it is very important to specify the location of these points.

Figure(3.1) showsthe liberation points(L_1, L_2, L_3,L_4 and L_5) for thetwo primary bodies (m_1 and m_2).

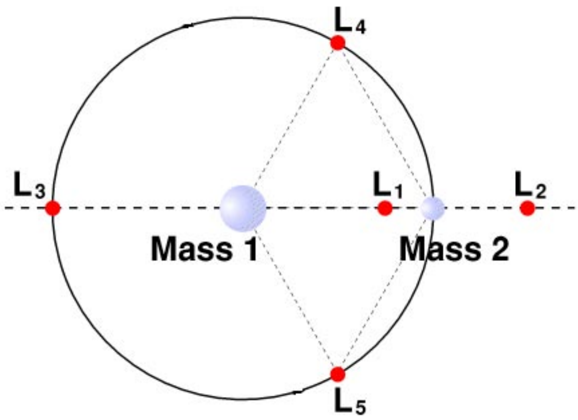

Figure3.1 : The five Lagrange Points associated with two primary bodies.

Some restrictions are considered to determine the locations of the liberation points, which are:

1- m_3 lies at any point of (L_1, L_2, L_3,L_4, L_5).

2- m_3 is very smaller than m_1 , m_2.

3- the third mass would have zero velocity and zero acceleration where would appear permanently at rest relative to m_1 and m_2 and the equilibrium points are defined when

$$\dot{x} = \dot{y} = \dot{z} = 0, \qquad (3.1)$$

$$\ddot{x} = \ddot{y} = \ddot{z} = 0\,. \qquad (3.2)$$

SubstitutingEqs.$^{(3.1)}$and$^{(3.2)}$into Eqs (2.1),(2.2) and (2.3) respectively, this yields

$$x = \frac{\mu_1}{r_1^3}(x + \mu) + \frac{\mu_2}{r_2^3}(x - (1 - \mu)), \qquad (3.3)$$

$$y = \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right]y, \qquad (3.4)$$

$$\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] z = 0\,. \qquad (3.5)$$

3.1.LocationofliberationpointsofL$_4$ and L$_5$

After some little algebraic calculations had been done to solve (3.3),(3.4)and(3.5) then it is found that

$$x = \frac{1}{2} - \mu, \qquad (3.1.1)$$

$$y = \pm\frac{\sqrt{3}}{2}. \qquad (3.1.2)$$

So, the coordinates ofL_4 and L_5 are being

$L_4\ (\frac{1}{2} - \mu, \frac{\sqrt{3}}{2})$ and

$L_5\ (\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2})$.

3.2- Location of liberation points of L_1,L_2 and L_3

L_1,L_2 and L_3

RecallEqs. (2.1), (2.2) and (2.3) with y = 0 as well as z = 0,then the three collinear points

(L_1 ,L_2 ,L_3) could be found from Eq.(3.3)

Now we can calculate(L_1 ,L_2 ,L_3) from these eqs.

1- ForL_1 L_1 lies between masses $m_1$and m_2 $m_1$and m_2 and it can be calcul

$$x-\frac{(1-\mu)}{(x+\mu)^2}+\frac{\mu}{\left(x-(1-\mu)\right)^2}=0. \quad (3.2.1)$$

2- ForL_2 lies outside the masse m_2

and it can be calculatedfrom nonlinear equation

$$x-\frac{(1-\mu)}{(x+\mu)^2}-\frac{\mu}{\left(x-(1-\mu)\right)^2}=0\,. \quad (3.2.2)$$

3- ForL_3 point lies on the negative

x-axis and it can be calculated from nonlin

$$x+\frac{(1-\mu)}{(x+\mu)^2}+\frac{\mu}{\left(x-(1-\mu)\right)^2}=0\,. \quad (3.2.3)$$

4. Jacobi Integral

Multiply Eq. (2.1) by$\dot{x}$, Eq. (2.2) by$\dot{y}$ and Eq. (2.3) by $\dot{z}$ to obtain

$$\ddot{x}\dot{x}-2\dot{x}\dot{y}-\dot{x}x=-\frac{\mu_1}{r_1^3}\dot{x}\,(x+\mu)-\frac{\mu_2}{r_2^3}\,\dot{x}\,(x-(1-\mu),$$

$$\ddot{y}\,\dot{y}+2\dot{x}\dot{y}-\dot{y}y=\;-\frac{\mu_1}{r_1^3}\dot{y}y-\frac{\mu_2}{r_2^3}\dot{y}y,$$

$$\ddot{z}\dot{z}=-\frac{\mu_1}{r_1^3}\dot{z}z-\frac{\mu_2}{r_2^3}z\dot{z}\,.$$

After some algebraic calculations, the integrationwas done to obtain the zero velocity curves (Moulton, 1970)

$$\frac{1}{2}v^2-\frac{1}{2}(x^2+y^2)-\frac{\mu_1}{r_1}-\frac{\mu_2}{r_2}=c \quad (4.1)$$

where

$\frac{1}{2}v^2$:is kinetic energy per unit mass relative to the rotating frame,

$-\frac{\mu_1}{r_1}$ and $-\frac{\mu_2}{r_2}$:arethe gravitational potential energy of the two primary masses respectively,

c: is called Jacobi integral, or Jacobi constant, sometimes called the integral of relative energy.

Equations (4.1) illustrates the zero velocity curves when$v=0$.

5. Stabilityof the liberation points

5.1 -Firstly at collinear points

To study the motion near any of the equilibrium point $L(x_0\,,y_0)$.L

$$x=x_0+\xi(5.1)y$$
$$=\;y_0+\eta$$

whereξ and η are the coordinate (5.2) and thePotential$V$is

$$V=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{r_1}+\;\frac{\mu}{r_2} \quad (5.3)$$

So,Vmay be expanded by Taylor series ... L_i,(i=1,...5) as

$$V=V(x_0,y_0)+V_x(x_0,y_0)\xi+V_y(x_0,y_0)\,\eta+\frac{1}{2!}V_{xx}(x_0,y_0)\xi^2+V_{xy}(x_0,y_0)\xi\eta+\frac{1}{2!}V_{yy}(x_0,y_0)\eta^2$$

where

$V_x$is first derivative of V with respect to x,

$V_y$is first derivative of V with respect to y,

$V_{xx}$is second derivative of V with respect to x,

$V_{yy}$is second derivative of V with respect to y,

The equation of motion of three body could be written in suitable form as

$$\ddot{x}-2\dot{y}=V_x\,, \quad (5.4)$$
$$\ddot{y}-2\dot{x}=V_y\,, \quad (5.5)$$
$$\ddot{z}=V_z\,. \quad (5.6)$$

SubstitutefromEqs.(5.1) and (5.2) into Eqs. (5.4), (5.5) and (5.6) respectively then it is found that

$$\ddot{\xi}-2\dot{y}=\xi V_{xx}\;+V_{xy}\eta\,, \quad (5.7)$$
$$\ddot{\eta}+2\dot{\xi}=\xi V_{xy}\;+V_{yy}\eta\,, \quad (5.8)$$
$$\ddot{\zeta}=\zeta V_{zz}\,. \quad (5.9)$$

Let

$$\xi=\alpha e^{\lambda t}\,, \quad (5.10)$$
$$\eta=\beta e^{\lambda t}\;\,. \quad (5.11)$$

Substitute fromEqs. (5.10) and (5.11) into Eqs (5.7), (5.8) and (5.9) respectively then it is found that,

$$(\lambda^2-V_{xx})\,\alpha=(2\lambda+V_{xy})\,\beta\,, \quad (5.12)$$

$$(2\lambda - V_{xy})\alpha = (\lambda^2 - V_{yy})\beta . \quad (5.13)$$

The characteristic equation becomes (Moulton, 1970):

$$\lambda^4 + (4 - V_{xx} - V_{yy})\lambda^2 + V_{xx}V_{yy} - V_{xy}^2 = 0 . \quad (5.14)$$

By solvingeq. (5.14) the roots of$\lambda$are obtained. If the roots obtained are pure nary numbers, then ξ and η are periodic and this stable periodic solution in the vicinity of x_0 and y_0 can be studied as:

1- If any of the λroots are real or complex number, then ξ and η increase with time so that the solution is unstable. This can be happened because the solution contains constants terms in the form of exponentials.

2- If the remaining exponentials are purely imaginaries. Then the solution is stable.

To obtain the expressions V_{xx}, V_{yy}, V_{xy}, V_{xz} and V_{yz} in terms of r_1, r_2 and μ, let

$$r_i^2 = (x - x_i)^2 + y^2 + z^2 , \qquad i = 1,2 ;$$

$$A = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3},$$

$$B = 3\left(\frac{1-\mu}{r_1^5} + \frac{\mu}{r_2^5}\right),$$

$$C = 3\left[\frac{1-\mu}{r_1^5}(x_0 - x_1) + \frac{\mu}{r_2^5}(x_0 - x_2)\right].$$

$$V_{xx} = 1 - A + 3(1-\mu)\frac{(x_0 - x_1)^2}{r_1^5} + 3\mu\frac{(x_0 - x_2)^2}{r_2^5}, \quad (5.15)$$

$$V_{yy} = 1 - A + By_0^2 , \quad (5.16)$$

$$V_{zz} = -A + Bz_0^2 , \quad (5.17)$$

while

$$V_{xy} = C\,y_0 , \quad V_{xz} = Cz_0 , \quad V_{yz} = By_0z_0 .$$

In the case of collinear points, $y_0 = z_0 = 0$, so that

$$r_i^2 = (x_0 - x_i)^2, \qquad \text{i.e.,} \qquad i = 1,2 \quad (5.18)$$

$$V_{xy} = V_{xz} = V_{yz} = 0.$$

Then, the equations of motion become

$$\ddot{\xi} - 2\dot{y} = \xi V_{xx} = \xi(1 + 2A), \quad (5.19)$$

$$\ddot{\eta} + 2\dot{\xi} = V_{yy}\eta = \eta(1 - A), \quad (5.20)$$

$$\ddot{\zeta} = \zeta V_{zz} = -A\zeta. \quad (5.21)$$

The Last equation is independent of the first twoEqs. and its solution is

$$\zeta = c_1 \sin t + c_2 \cos t.$$

Therefore the motion parallel to the z-axis for small displacement is periodic with period 2π.Applying the values of V_{xx}, V_{xy} and V_{yy} V_{yy} in equation (5.14) yields

$$\lambda^4 + (2 - A)\lambda^2 + (1 + A - 2A^2) = 0 \quad (5.22)$$

Now there are three values of A corresponding to the three Lagrangian points L_1, L_2, L_3 obtained from equations (3.2.1), (3.2.2) and (3.2.3) respectively. It can be shown that the values of L_1, L_2 and L_3 the next condition is verified.

$$1 + A - 2A^2 < 0$$

While values of μ up to its limit 1/2.Then, the four roots of equation (5.22)consist of two real roots, numerically equal but opposite in sign, and two conjugate pure imaginary roots. Then the solution for the straight-line case is unstable and the orbit becomes spiral.

5.2-Secondly triangular points:

The coordinate of the triangular equilibrium points L_4, L_5 are

$$x_0 = \frac{1}{2} - \mu \qquad , y_0 = \pm\frac{\sqrt{3}}{2}$$

AtL_4 from Eqs (5.15),(5.16),(5.17) and(5.22) it is found that

$$V_{xx} = {}^3\!/_4, V_{yy} = {}^9\!/_4, V_{zz} = -1 ,$$

$$V_{xy} = \frac{3\sqrt{3}}{4}(1 - 2\mu) , \qquad V_{xz} = V_{yz} V_{yz} = 0 . \quad (5.23)$$

The equations of motion at L_4 become

$$\ddot{\xi} - 2\dot{\eta} = {}^{3}/_{4}\,\xi + \frac{3\sqrt{3}}{4}(1-2\mu)\xi, \quad (5.24)$$

$$\ddot{\eta} + 2\dot{\xi} = \frac{3\sqrt{3}}{4}(1-2\mu)\xi \;+\frac{9}{4}, \quad (5.25)$$

$$\ddot{\zeta} = \zeta V_{zz} = -\zeta\ .$$

The Last equation is independent of the first two and its solution is

$$\zeta = c_1 \sin t + c_2 \cos t\ .$$

so that the motion parallel to the z-axis for small displacement and the solution is periodic withperiod 2π.

As the same way to determine the characteristic equation for collinear points it is found that thecharacteristic equation for L_4 becomes

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-2\mu)=0. \quad (5.26)$$

If $\mu \leq$ ½ and if $1 - 27\mu(1-2\mu) \geq 0$ the roots are pure imaginary.

The inequality may be written as

$$1 - 27\mu(1-2\mu) = \varepsilon,$$

where ε is a positive quantity whose limit is zero.

The solution of this equation is

$$\mu = \frac{1}{2} \pm \sqrt{\frac{23+4\varepsilon}{108}}\ .$$

Since μ represents the mass ratio, which is less than ½the negative sign must be tak

en at the limit, $\varepsilon = 0 \;\rightarrow \mu = 0.0385$.0

Therefore if $\mu < 0.0385$ the roots become pure imaginaries and the motion of the particle displaced from the equilibrium point is oscillatory in form,hence the particle will remain in the vicinity ofequilibrium point and the motion become stable.

If $\mu > 0.0385$ the roots become complex and the orbits become spiral.

The spiral orbits asymptotically approach the triangular libration points or depart from them;therefore, the motionbecomes unstable.

6. RESULTS AND CONCLUSION

A code was constructed by MATHEMATICA language, and applied on the ten binary asteroids to obtain liberation points,mass ratio and drawing contour plot of zero velocity curves. Selected two binary asteroids One of them its triangular points stable and another unstable at L_4 and L_5.By usingthese relations to determine mass ratio from half- diameter ofeach mass binary asteroids as

$$\mu = \frac{m_2}{m_1 + m_2}$$

$$= \frac{\frac{4}{3}\pi\rho {R^3}_2}{\frac{4}{3}\pi\rho({R^3}_1+{R^3}_2)}$$

$$= \frac{{R^3}_2}{{R^3}_1+{R^3}_2}\ .$$

Tables (6.1) and (6.2) show the results for the given data of two asteroid diameters(D1 and D2)and the distance between them(R_b), which are usedby the code to determine the liberation points(L_1, L_2, L_3,L_4 and L_5) for the ten binary asteroids.Figures (6.1) and (6.2)display the contours of zero velocity curves for stable binary 1996FG3 at $\mu = 0.02829$ and for unstablebinary 1999 DJ4 at μ=0.1111 respectively.

Table 6.1: collinear points and stability for binary system.
http://echo.jpl.nasa.gov/ lance/binary.neas.html.

No	Binary	D_1 (km)	D_2 (km)	R_b (km)	μ	L_1 (km)	L_2 (km)	L_3 (km)	Stability For L_1, L_2, L_3
1	VH 1991	1.1	44.	3.2	0601.	2.212	3.959	-3.28	unstable
2	AW1 1994	1	49.	2.3	10526.	1.337	2.901	-2.40	unstable
3	FG3 1996	1.5	465.	2.6	02892.	2.009	3.118	-2.63	unstable
4	PG 1998	9.	27.	1.5	02629.	1.127	1.79	-1.51	unstable
5	DJ4 1999	35.	175.	8.	11111.	47.	9093.	836.-	unstable
6	KW4 1999	1.5	57.	2.54	05201.	1.804	3.124	-5.595	unstable
7	DP107 2000	8.	328.	2.6	06447.	1.77	2.955	-2.699	unstable
8	UG11 2000	26.	156.	4.	17763.	189.	508.	429.-	unstable
9	SL9 2001	8.	224.	1.4	02148.	1.117	1.657	-1.412	unstable
10	CE26 2002	3	21.	5.1	00034.	4.854	5.349	-5.1007	unstable

Table 6.2: triangular points and stabilities for binary system.

No	Binary	L_4 (x, y)		L_5 (x, y)		Stability L_4, L_5 For
1	VH 1991	1.407	2.771	1.407	-2.771	Unstable
2	AW1 1994	0.908	1.991	0.908	-1.991	Unstable
3	FG3 1996	1.285	2.251	1.285	-2.251	Stable
4	PG 1998	0.711	1.299	0.711	-1.299	Stable
5	DJ4 1999	0.311	0.692	0.311	-0.692	unstable
6	KW4 1999	1.139	2.199	1.139	-2.199	unstable
7	DP107 2000	1.132	2.251	1.132	-2.251	unstable
8	UG11 2000	0.129	0.346	0.129	-0.346	unstable
9	SL9 2001	0.670	1.212	0.670	-1.212	stable
10	CE26 2002	2.584	4.42	2.584	-4.42	stable

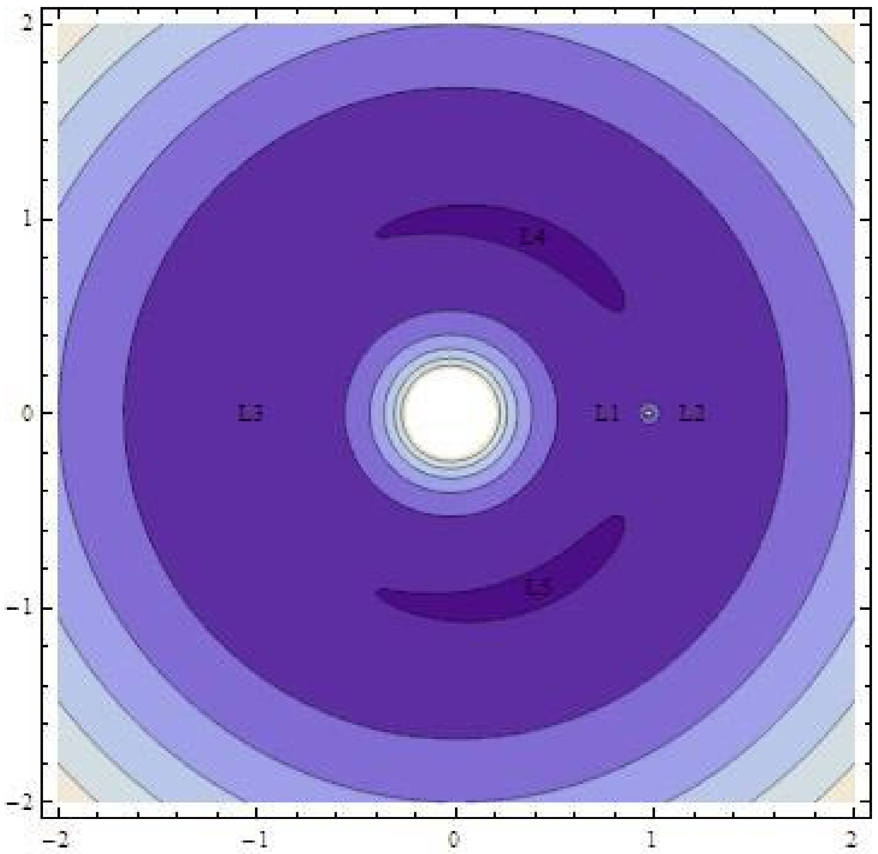

Figure 6.1: Contour plotfor 1996 FG3 binary system with $\mu = 0.02829$

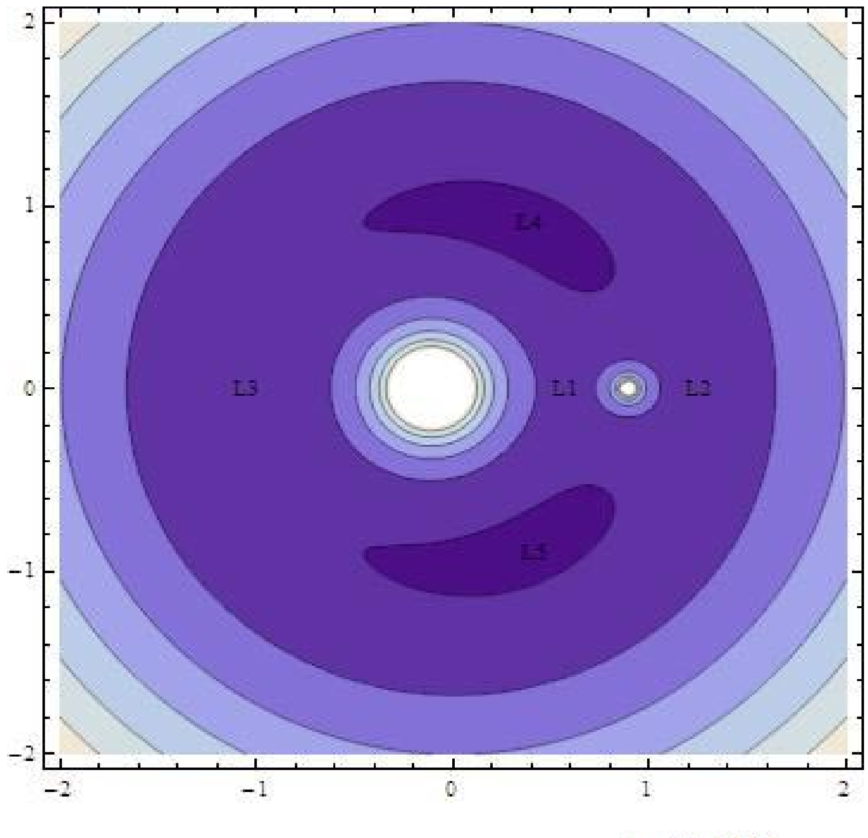

Figure 6.2: Contour plotfor1999 DJ4 binary systemwith $\mu = 0.1111$

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