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On Binomial Transform of the Generalized Fifth Order Pell Sequence

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we define the binomial transform of the generalized fifth order Pell sequence and as special cases, the binomial transform of the fifth order Pell and fifth order Pell-Lucas sequences will be introduced. We investigate their properties in details. We present Binet's formulas, generating functions, Simson formulas, recurrence properties, and the summation formulas for these binomial transforms. Moreover, we give some identities and matrices related with these binomial transforms.

Keywords: Binomial transform; fifth order Pell Sequence; fifth order Pell numbers; binomial transform of fifth order Pell Sequence; binomial transform of fifth order Pell-Lucas sequence.

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1 INTRODUCTION AND PRELIMINARIES

In this paper, we introduce the binomial transform of the generalized fifth order Pell sequence and we investigate, in detail, two special cases which we call them the binomial transform of the fifth

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order Pell and fifth order Pell-Lucas sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized (r, s, t, u, v) sequence (generalized Pentanacci) sequence.

The generalized (r, s, t, u, v) sequence (the generalized Pentanacci sequence or 5-step Fibonacci sequence)

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$$

is defined by the fifth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \tag{1.1}$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1,2,3,4,5]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

As $\{W_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0 \tag{1.2}$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda$. Note that we have the following identities:

$$\begin{aligned}
\alpha + \beta + \gamma + \delta + \lambda &= r, \\
\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \alpha\delta + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta &= -s, \\
\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\gamma + \alpha\gamma\delta + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta &= t, \\
\alpha\beta\lambda\gamma + \alpha\beta\lambda\delta + \alpha\beta\gamma\delta + \alpha\lambda\gamma\delta + \beta\lambda\gamma\delta &= -u \\
\alpha\beta\gamma\delta\lambda &= v.
\end{aligned}$$

Generalized Pentanacci numbers can be expressed, for all integers n, using Binet's formula.

Theorem 1.1. *[4, Theorem 1.] (Binet's formula of generalized* (r, s, t, u, v) *numbers (generalized Pentanacci numbers))*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned} \tag{1.3}$$

where

$$\begin{aligned}
p_1 &= W_4 - (\beta + \gamma + \delta + \lambda)W_3 + (\beta\lambda + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta)W_2 - (\beta\lambda\gamma + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta)W_1 + (\beta\lambda\gamma\delta)W_0, \\
p_2 &= W_4 - (\alpha + \gamma + \delta + \lambda)W_3 + (\alpha\lambda + \alpha\gamma + \alpha\delta + \lambda\gamma + \lambda\delta + \gamma\delta)W_2 - (\alpha\lambda\gamma + \alpha\lambda\delta + \alpha\gamma\delta + \lambda\gamma\delta)W_1 + (\alpha\lambda\gamma\delta)W_0, \\
p_3 &= W_4 - (\alpha + \beta + \delta + \lambda)W_3 + (\alpha\beta + \alpha\lambda + \beta\lambda + \alpha\delta + \beta\delta + \lambda\delta)W_2 - (\alpha\beta\lambda + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\delta)W_1 + (\alpha\beta\lambda\delta)W_0, \\
p_4 &= W_4 - (\alpha + \beta + \gamma + \lambda)W_3 + (\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \beta\gamma + \lambda\gamma)W_2 - (\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \beta\lambda\gamma)W_1 + (\alpha\beta\lambda\gamma)W_0, \\
p_5 &= W_4 - (\alpha + \beta + \gamma + \delta)W_3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)W_2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)W_1 + (\alpha\beta\gamma\delta)W_0.
\end{aligned}$$

Usually, it is customary to choose r, s, t, u, v so that the Equ. (1.2) has at least one real (say α) solutions.

(1.3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)},\\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)},\\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)},\\
A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)},\\
A_5 &= \frac{p_5}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}.
\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.2. *[4, Lemma 2.] Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r, s, t, u, v) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3+(W_4-rW_3-sW_2-tW_1-uW_0)x^4}{1-rx-sx^2-tx^3-ux^4-vx^5}. \tag{1.4}$$

We next find Binet formula of generalized (r, s, t, u, v) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.3. *[4, Theorem 3.] (Binet's formula of generalized (r, s, t, u, v) numbers)*

$$\begin{aligned}
W_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\
&\quad +\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{q_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}
\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}
q_1 &= W_0\alpha^4+(W_1-rW_0)\alpha^3+(W_2-rW_1-sW_0)\alpha^2+(W_3-rW_2-sW_1-tW_0)\alpha+(W_4-rW_3-sW_2-tW_1-vW_0),\\
q_2 &= W_0\beta^4+(W_1-rW_0)\beta^3+(W_2-rW_1-sW_0)\beta^2+(W_3-rW_2-sW_1-tW_0)\beta+(W_4-rW_3-sW_2-tW_1-vW_0),\\
q_3 &= W_0\gamma^4+(W_1-rW_0)\gamma^3+(W_2-rW_1-sW_0)\gamma^2+(W_3-rW_2-sW_1-tW_0)\gamma+(W_4-rW_3-sW_2-tW_1-vW_0),\\
q_4 &= W_0\delta^4+(W_1-rW_0)\delta^3+(W_2-rW_1-sW_0)\delta^2+(W_3-rW_2-sW_1-tW_0)\delta+(W_4-rW_3-s3W_2-tW_1-vW_0),\\
q_5 &= W_0\lambda^4+(W_1-rW_0)\lambda^3+(W_2-rW_1-sW_0)\lambda^2+(W_3-rW_2-sW_1-tW_0)\lambda+(W_4-rW_3-sW_2-tW_1-vW_0).
\end{aligned}$$

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix} \tag{1.6}$$

For matrix formulation (1.6), see [6]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ r & s & t & u & v \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}.$$

Next, we consider two special cases of the generalized (r,s,t,u,v) sequence $\{W_n\}$ which we call them (r,s,t,u,v) and Lucas (r,s,t,u,v) sequences. (r,s,t,u,v) sequence$\{G_n\}_{n\geq 0}$ and Lucas (r,s,t,u,v) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fifth-order recurrence relations

$$\begin{aligned} G_{n+5} &= rG_{n+4}+sG_{n+3}+tG_{n+2}+uG_{n+1}+vG_n, \quad (1.7)\\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, G_4=r^3+2sr+t,\\ H_{n+5} &= rH_{n+4}+sH_{n+3}+tH_{n+2}+uH_{n+1}+vH_n, \quad (1.8)\\ H_0 &= 5, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t, H_4=r^4+4r^2s+4tr+2s^2+4u. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{u}{v}G_{-(n-1)}-\frac{t}{v}G_{-(n-2)}-\frac{s}{v}G_{-(n-3)}-\frac{r}{v}G_{-(n-4)}+\frac{1}{v}G_{-(n-5)},\\ H_{-n} &= -\frac{u}{v}H_{-(n-1)}-\frac{t}{v}H_{-(n-2)}-\frac{s}{v}H_{-(n-3)}-\frac{r}{v}H_{-(n-4)}+\frac{1}{v}H_{-(n-5)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.7) and (1.8) hold for all integers $n.$

For more details on the generalized (r,s,t,u,v) numbers, see Soykan [4].

Some special cases of (r,s,t,u,v) sequence $\{G_n(0,1,r,r^2+s,r^3+2sr+t;r,s,t,u,v)\}$ and Lucas (r,s,t,u,v) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t,r^4+4r^2s+4tr+2s^2+4u;r,s,t,u,v)\}$ are as follows:

1. $G_n(0,1,1,2,4;1,1,1,1,1)=P_n$, Pentanacci sequence,
2. $H_n(5,1,3,7,15;1,1,1,1,1)=Q_n$, Pentanacci-Lucas sequence,
3. $G_n(0,1,2,5,13;2,1,1,1,1)=P_n$, fifth-order Pell sequence,
4. $H_n(5,2,6,17,46;2,1,1,1,1)=Q_n$, fifth-order Pell-Lucas sequence,

For all integers n, (r,s,t,u,v) and Lucas (r,s,t,u,v) numbers (using initial conditions in (1.3) or (1.5)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}\\ &\quad +\frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},\\ H_n &= \alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n, \end{aligned}$$

respectively.

Lemma 1.2 gives the following results as particular examples (generating functions of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers).

Corollary 1.4. *Generating functions of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3-ux^4-vx^5},\\ \sum_{n=0}^{\infty} H_n x^n &= \frac{5-4rx-3sx^2-2tx^3-ux^4}{1-rx-sx^2-tx^3-ux^4-vx^5}, \end{aligned}$$

respectively.

The following theorem shows that the generalized Pentanacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.5. *[5, Theorem 1.] For $n \in \mathbb{Z}$, for the generalized Pentanacci sequence (or generalized (r,s,t,u,v)-sequence or 5-step Fibonacci sequence) we have the following:we have*
$W_{-n} = \frac{1}{24}v^{-n}(W_0H_n^4 - 4W_nH_n^3 + 3W_0H_{2n}^2 + 12H_n^2W_{2n} - 6W_0H_n^2H_{2n} - 6W_0H_{4n} - 8W_nH_{3n} - 12H_{2n}W_{2n} - 24H_nW_{3n} + 24W_{4n} + 8W_0H_nH_{3n} + 12W_nH_nH_{2n})$
$= v^{-n}(W_{4n} - H_nW_{3n} + \frac{1}{2}(H_n^2 - H_{2n})W_{2n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_n + \frac{1}{24}(H_n^4 + 3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n)W_0)$.

Using Theorem 1.5, we have the following corollary, see Soykan [5, Corollary 4].

Corollary 1.6. *For $n \in \mathbb{Z}$, we have*

$$H_{-n} = \frac{1}{24}v^{-n}(H_n^4 + 3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n).$$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 5$ in Theorem 1.5:

$$\begin{aligned} G_{-n} &= v^{-n}(G_{4n} - H_nG_{3n} + \frac{1}{2}(H_n^2 - H_{2n})G_{2n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)G_n), \\ H_{-n} &= \frac{1}{24}v^{-n}(H_n^4 + 3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n), \end{aligned}$$

respectively.

Next, we consider the case $r = 2,\ s = 1, t = 1, u = 1, v = 1$ and in this case we write $V_n = W_n$. A generalized fifth order Pell sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4)\}_{n\geq 0}$ is defined by the fifth-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} \tag{1.9}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} - 2V_{-(n-4)} + V_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.9) holds for all integer n. For more information on the generalized fifth order Pell numbers, see Soykan [7].

The first few generalized fifth order Pell numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized fifth order Pell numbers

n	V_n	V_{-n}
0	V_0	V_0
1	V_1	$-V_0 - V_1 - V_2 - 2 \times V_3 + V_4$
2	V_2	$-V_4 + 3V_3 - V_2$
3	V_3	$-V_3 + 3V_2 - V_1$
4	V_4	$-V_2 + 3V_1 - V_0$
5	$2V_4 + V_3 + V_2 + V_1 + V_0$	$-V_4 + 2V_3 + V_2 + 4V_0$
6	$5V_4 + 3V_3 + 3V_2 + 3V_1 + 2V_0$	$4V_4 - 9V_3 - 2V_2 - 3V_1 - 4V_0$
7	$13V_4 + 8V_3 + 8V_2 + 7V_1 + 5V_0$	$-4V_4 + 12V_3 - 5V_2 + 2V_1 + V_0$
8	$34V_4 + 21V_3 + 20V_2 + 18V_1 + 13V_0$	$V_4 - 6V_3 + 11V_2 - 6V_1 + V_0$
9	$89V_4 + 54V_3 + 52V_2 + 47V_1 + 34V_0$	$V_4 - V_3 - 7V_2 + 10V_1 - 7V_0$
10	$232V_4 + 141V_3 + 136V_2 + 123V_1 + 89V_0$	$-7V_4 + 15V_3 + 6V_2 + 17V_0$

(1.3) can be used to obtain Binet's formula of generalized fifth order Pell numbers. Generalized fifth order Pell numbers can be expressed, for all integers $n,$ using Binet's formula

$$\begin{aligned} V_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad +\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \end{aligned}$$

where

$$\begin{aligned} p_1 &= V_4-(\beta+\gamma+\delta+\lambda)V_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_2-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_1+(\beta\lambda\gamma\delta)V_0, \\ p_2 &= V_4-(\alpha+\gamma+\delta+\lambda)V_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)V_2-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)V_1+(\alpha\lambda\gamma\delta)V_0, \\ p_3 &= V_4-(\alpha+\beta+\delta+\lambda)V_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)V_2-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)V_1+(\alpha\beta\lambda\delta)V_0, \\ p_4 &= V_4-(\alpha+\beta+\gamma+\lambda)V_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)V_2-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)V_1+(\alpha\beta\lambda\gamma)V_0, \\ p_5 &= V_4-(\alpha+\beta+\gamma+\delta)V_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)V_2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)V_1+(\alpha\beta\gamma\delta)V_0. \end{aligned}$$

Here, $\alpha,\beta,\gamma,\delta$ and λ are the roots of the equation

$$x^5-2x^4-x^3-x^2-x-1=0. \tag{1.10}$$

Moreover, the approximate value of $\alpha,\beta,\gamma,\delta$ and λ are given by

$$\begin{aligned} \alpha &= 2.6083299 \\ \beta &= 0.282\,69438-0.794\,69421i \\ \gamma &= 0.282\,69438+0.794\,69421i \\ \delta &= -0.586\,85934-0.440\,99162i \\ \lambda &= -0.586\,85934+0.440\,99162i \end{aligned}$$

Note that we have the following identities:

$$\begin{aligned} \alpha+\beta+\gamma+\delta+\lambda &= 2, \\ \alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -1, \\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= 1, \\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -1 \\ \alpha\beta\gamma\delta\lambda &= 1. \end{aligned}$$

Now we consider two special case of the sequence $\{V_n\}$. Fifth-order Pell sequence $\{P_n\}_{n\geq0}$ and fifth-order Pell-Lucas sequence $\{Q_n\}_{n\geq0}$ are defined, respectively, by the fifth-order recurrence relations

$$P_{n+5}=2P_{n+4}+P_{n+3}+P_{n+2}+P_{n+1}+P_n, \quad P_0=0,P_1=1,P_2=2,P_3=5,P_4=13, \tag{1.11}$$

and

$$Q_{n+5}=2Q_{n+4}+Q_{n+3}+Q_{n+2}+Q_{n+1}+Q_n, \quad Q_0=4,Q_1=2,Q_2=6,Q_3=17,Q_4=46. \tag{1.12}$$

The sequences $\{P_n\}_{n\geq0}$ and $\{Q_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$\begin{aligned} P_{-n} &= -P_{-(n-1)}-P_{-(n-2)}-P_{-(n-3)}-2P_{-(n-4)}+P_{-(n-5)}, \\ Q_{-n} &= -Q_{-(n-1)}-Q_{-(n-2)}-Q_{-(n-3)}-2Q_{-(n-4)}+Q_{-(n-5)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.11) and (1.12) hold for all integer $n.$

Next, we present the first few values of the fifth order Pell and fifth order Pell-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few fifth order Pell and fifth order Pell-Lucas Numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12
P_n	0	1	2	5	13	34	89	232	605	1578	4116	10736	28003
P_{-n}	0	0	0	0	1	-1	0	0	-1	4	-4	1	1
Q_n	5	2	6	17	46	122	315	821	2142	5588	14576	38018	99163
Q_{-n}	5	-1	-1	-1	-5	14	-7	-1	3	-28	54	-34	1

For all integers $n,$ usual fifth order Pell and fifth order Pell-Lucas numbers can be expressed using Binet's formulas

$$\begin{aligned} P_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} \\ &+\frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}+\frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)} \end{aligned}$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n$$

respectively, see [7, Corollary 3.2.].

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1.7. *[7, Lemma 2.1.]Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized fifth-order Pell sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-2V_0)x+(V_2-2V_1-V_0)x^2+(V_3-2V_2-V_1-V_0)x^3+(V_4-2V_3-V_2-V_1-V_0)x^4}{(1-2x-x^2-x^3-x^4-x^5)}. \quad (1.13)$$

The previous Lemma gives the following results as particular examples: generating function of the fifth order Pell sequence P_n is

$$f_{P_n}(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2-x^3-x^4-x^5}$$

and generating function of the fifth order Pell-Lucas sequence Q_n is

$$f_{Q_n}(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{5-8x-3x^2-2x^3-x^4}{1-2x-x^2-x^3-x^4-x^5},$$

see [7, Corollary 2.2.].

2 BINOMIAL TRANSFORM OF THE GENERALIZED FIFTH ORDER PELL SEQUENCE V_n

In [8, p. 137], Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ may be defined by the rule

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} a_i, \text{ with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} a_i, \text{ with inversion } a_n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} \hat{a}_i.$$

For more information on binomial transform, see, for example, [9,10,11,12] and references therein. For recent works on binomial transform of well-known sequences, see for example, [13,14,15,16,17, 18,19,20,21,22,23,24,25].

In this section, we define the binomial transform of the generalized fifth order Pell sequence V_n and as special cases the binomial transform of the fifth order Pell and fifth order Pell-Lucas sequences will be introduced.

Definition 2.1. The binomial transform of the generalized fifth order Pell sequence V_n is defined by

$$b_n = \widehat{V}_n = \sum_{i=0}^{n} \binom{n}{i} V_i.$$

The few terms of b_n are

$$\begin{aligned}
b_0 &= \sum_{i=0}^{0} \binom{0}{i} V_i = V_0, \\
b_1 &= \sum_{i=0}^{1} \binom{1}{i} V_i = V_0 + V_1, \\
b_2 &= \sum_{i=0}^{2} \binom{2}{i} V_i = V_0 + 2V_1 + V_2, \\
b_3 &= \sum_{i=0}^{3} \binom{3}{i} V_i = V_0 + 3V_1 + 3V_2 + V_3, \\
b_4 &= \sum_{i=0}^{4} \binom{4}{i} V_i = V_0 + 4V_1 + 6V_2 + 4V_3 + V_4.
\end{aligned}$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the fifth order Pell and fifth order Pell-Lucas sequences are defined as follows: The binomial transform of the fifth order Pell sequence P_n is

$$\widehat{P}_n = \sum_{i=0}^{n} \binom{n}{i} P_i,$$

and the binomial transform of the fifth order Pell-Lucas sequence Q_n is

$$\widehat{Q}_n = \sum_{i=0}^{n} \binom{n}{i} Q_i.$$

Lemma 2.1. *For $n \geq 0$, the binomial transform of the generalized fifth order Pell sequence V_n satisfies the following relation:*

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned} b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\ &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \\ &= V_0 + \sum_{i=1}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}). \end{aligned}$$

This completes the proof. □

Remark 2.1. From the last Lemma, we see that

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized fifth order Pell sequence.The following theorem gives recurrent relations of the binomial transform of the generalized fifth order Pell sequence.

Theorem 2.2. *For $n \geq 0$, the binomial transform of the generalized fifth order Pell sequence V_n satisfies the following recurrence relation:*

$$b_{n+5} = 7b_{n+4} - 17b_{n+3} + 20b_{n+2} - 11b_{n+1} + 3b_n. \quad (2.1)$$

Proof. To show (2.1), writing

$$b_{n+5} = r_1 \times b_{n+4} + s_1 \times b_{n+3} + t_1 \times b_{n+2} + u_1 \times b_{n+1} + v_1 \times b_n$$

and taking the values $n = 0, 1, 2, 3, 4$ and then solving the system of equations

$$\begin{aligned} b_5 &= r_1 \times b_4 + s_1 \times b_3 + t_1 \times b_2 + u_1 \times b_1 + v_1 \times b_0 \\ b_6 &= r_1 \times b_5 + s_1 \times b_4 + t_1 \times b_3 + u_1 \times b_2 + v_1 \times b_1 \\ b_7 &= r_1 \times b_6 + s_1 \times b_5 + t_1 \times b_4 + u_1 \times b_3 + v_1 \times b_2 \\ b_8 &= r_1 \times b_7 + s_1 \times b_6 + t_1 \times b_5 + u_1 \times b_4 + v_1 \times b_3 \\ b_9 &= r_1 \times b_8 + s_1 \times b_7 + t_1 \times b_6 + u_1 \times b_5 + v_1 \times b_4 \end{aligned}$$

we find that $r_1 = 7, s_1 = -17, t_1 = 20, u_1 = -11, v_1 = 3$. □

The sequence $\{b_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = \frac{11}{3}b_{-(n-1)} - \frac{20}{3}b_{-(n-2)} + \frac{17}{3}b_{-(n-3)} - \frac{7}{3}b_{-(n-4)} + \frac{1}{3}b_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

Note that the recurence relation (2.1) is independent from initial values. So,

$$\begin{aligned} \widehat{P}_{n+5} &= 7\widehat{P}_{n+4} - 17\widehat{P}_{n+3} + 20\widehat{P}_{n+2} - 11\widehat{P}_{n+1} + 3\widehat{P}_n, \\ \widehat{Q}_{n+5} &= 7\widehat{Q}_{n+4} - 17\widehat{Q}_{n+3} + 20\widehat{Q}_{n+2} - 11\widehat{Q}_{n+1} + 3\widehat{Q}_n. \end{aligned}$$

The first few terms of the binomial transform of the generalized fifth order Pell sequence with positive subscript and negative subscript are given in the following Table 3.

Table 3. A few binomial transform (terms) of the generalized fifth order Pell sequence

n	b_n	b_{-n}
0	V_0	V_0
1	$V_0 + V_1$	$\frac{1}{3}(2V_0 - 3V_1 + 2V_2 - 3V_3 + V_4)$
2	$V_0 + 2V_1 + V_2$	$-\frac{1}{9}(5V_0 + 15V_1 - 10V_2 + 30V_3 - 11V_4)$
3	$V_0 + 3V_1 + 3V_2 + V_3$	$-\frac{1}{27}(76V_0 + 30V_1 + V_2 + 150V_3 - 61V_4)$
4	$V_0 + 4V_1 + 6V_2 + 4V_3 + V_4$	$-\frac{1}{81}(392V_0 - 138V_1 + 305V_2 + 309V_3 - 164V_4)$
5	$2V_0 + 6V_1 + 11V_2 + 11V_3 + 7V_4$	$-\frac{1}{243}(814V_0 - 1590V_1 + 2143V_2 - 1578V_3 + 362V_4)$
6	$9V_0 + 15V_1 + 24V_2 + 29V_3 + 32V_4$	$\frac{1}{729}(4045V_0 + 7212V_1 - 7154V_2 + 18\,375V_3 - 6487V_4)$
7	$41V_0 + 56V_1 + 71V_2 + 85V_3 + 125V_4$	$\frac{1}{2187}(47318V_0 + 9501V_1 + 4220V_2 + 86088V_3 - 35183V_4)$

The first few terms of the binomial transform numbers of the fifth order Pell and fifth order Pell-Lucas sequences with positive subscript and negative subscript are given in the following Table 4.

Table 4. A few binomial transform (terms)

n	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{P}_n$	0	1	4	14	49	174	624	2248	8111	29274	105649	381249
$\widehat{P}_{-n}$		$-\frac{1}{3}$	$-\frac{2}{9}$	$\frac{11}{27}$	$\frac{115}{81}$	$\frac{488}{243}$	$\frac{448}{729}$	$-\frac{8998}{2187}$	$-\frac{72038}{6561}$	$-\frac{266911}{19683}$	$-\frac{35579}{59049}$	$\frac{6338387}{177147}$
$\widehat{Q}_n$	5	7	15	46	163	597	2184	7938	28723	103717	374255	1350290
$\widehat{Q}_{-n}$		$\frac{11}{3}$	$\frac{1}{9}$	$-\frac{190}{27}$	$-\frac{1223}{81}$	$-\frac{3574}{243}$	$\frac{5698}{729}$	$\frac{125990}{2187}$	$\frac{712945}{6561}$	$\frac{1731212}{19683}$	$-\frac{5823314}{59049}$	$-\frac{82199161}{177147}$

(1.3) can be used to obtain Binet's formula of the binomial transform of generalized fifth order Pell numbers. Binet's formula of the binomial transform of generalized fifth order Pell numbers can be given as

$$\begin{aligned} b_n &= \frac{C_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)} + \frac{C_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)} \\ &\quad + \frac{C_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)} + \frac{C_4\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)} \\ &\quad + \frac{C_5\theta_5^n}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
C_1 &= b_4-(\theta_2+\theta_3+\theta_4+\theta_5)b_3+(\theta_2\theta_5+\theta_2\theta_3+\theta_5\theta_3+\theta_2\theta_4+\theta_5\theta_4+\theta_3\theta_4)b_2 \\
&\quad -(\theta_2\theta_5\theta_3+\theta_2\theta_5\theta_4+\theta_2\theta_3\theta_4+\theta_5\theta_3\theta_4)b_1+(\theta_2\theta_5\theta_3\theta_4)b_0, \\
C_2 &= b_4-(\theta_1+\theta_3+\theta_4+\theta_5)b_3+(\theta_1\theta_5+\theta_1\theta_3+\theta_1\theta_4+\theta_5\theta_3+\theta_5\theta_4+\theta_3\theta_4)b_2 \\
&\quad -(\theta_1\theta_5\theta_3+\theta_1\theta_5\theta_4+\theta_1\theta_3\theta_4+\theta_5\theta_3\theta_4)b_1+(\theta_1\theta_5\theta_3\theta_4)b_0, \\
C_3 &= b_4-(\theta_1+\theta_2+\theta_4+\theta_5)b_3+(\theta_1\theta_2+\theta_1\theta_5+\theta_2\theta_5+\theta_1\theta_4+\theta_2\theta_4+\theta_5\theta_4)b_2 \\
&\quad -(\theta_1\theta_2\theta_5+\theta_1\theta_2\theta_4+\theta_1\theta_5\theta_4+\theta_2\theta_5\theta_4)b_1+(\theta_1\theta_2\theta_5\theta_4)b_0, \\
C_4 &= b_4-(\theta_1+\theta_2+\theta_3+\theta_5)b_3+(\theta_1\theta_2+\theta_1\theta_5+\theta_1\theta_3+\theta_2\theta_5+\theta_2\theta_3+\theta_5\theta_3)b_2 \\
&\quad -(\theta_1\theta_2\theta_5+\theta_1\theta_2\theta_3+\theta_1\theta_5\theta_3+\theta_2\theta_5\theta_3)b_1+(\theta_1\theta_2\theta_5\theta_3)b_0, \\
C_5 &= b_4-(\theta_1+\theta_2+\theta_3+\theta_4)b_3+(\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4)b_2 \\
&\quad -(\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4)b_1+(\theta_1\theta_2\theta_3\theta_4)b_0.
\end{aligned}$$

Here, $\theta_1, \theta_2, \theta_3, \theta_4$ and θ_5 are the roots of the equation

$$x^5 - 7x^4 + 17x^3 - 20x^2 + 11x - 3 = 0.$$

Moreover, the approximate value of $\theta_1, \theta_2, \theta_3, \theta_4$ and θ_5 are given by

$$\begin{aligned}
\theta_1 &= 3.60832992251682 \\
\theta_2 &= 1.28269437867436 + 0.794694205695784i \\
\theta_3 &= 1.28269437867436 - 0.794694205695784i \\
\theta_4 &= 0.413140660067231 + 0.440991619401373i \\
\theta_5 &= 0.413140660067231 - 0.440991619401373i
\end{aligned}$$

Note that

$$\begin{aligned}
\theta_1+\theta_2+\theta_3+\theta_4+\theta_5 &= 7 \\
\theta_1\theta_2+\theta_1\theta_5+\theta_1\theta_3+\theta_2\theta_5+\theta_1\theta_4+\theta_2\theta_3+\theta_5\theta_3+\theta_2\theta_4+\theta_5\theta_4+\theta_3\theta_4 &= 17 \\
\theta_1\theta_2\theta_5+\theta_1\theta_2\theta_3+\theta_1\theta_5\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_5\theta_4+\theta_2\theta_5\theta_3+\theta_1\theta_3\theta_4+\theta_2\theta_5\theta_4+\theta_2\theta_3\theta_4+\theta_5\theta_3\theta_4 &= 20 \\
\theta_1\theta_2\theta_5\theta_3+\theta_1\theta_2\theta_5\theta_4+\theta_1\theta_2\theta_3\theta_4+\theta_1\theta_5\theta_3\theta_4+\theta_2\theta_5\theta_3\theta_4 &= 11 \\
\theta_1\theta_2\theta_3\theta_4\theta_5 &= 3
\end{aligned}$$

For all integers n, (Binet's formulas of) binomial transforms of fifth order Pell and fifth order Pell-Lucas numbers (using initial conditions in (2.2)) can be expressed using Binet's formulas as

$$\begin{aligned}
\widehat{P}_n &= \frac{(\theta_1-1)^3\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)}+\frac{(\theta_2-1)^3\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)} \\
&\quad +\frac{(\theta_3-1)^3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)}+\frac{(\theta_4-1)^3\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)} \\
&\quad +\frac{(\theta_5-1)^3\theta_5^n}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)},
\end{aligned}$$

and

$$\widehat{Q}_n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n + \theta_5^n,$$

respectively.

3 GENERATING FUNCTIONS AND OBTAINING BINET FORMULA OF BINOMIAL TRANSFORM FROM GENERATING FUNCTION

The generating function of the binomial transform of the generalized fifth order Pell sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized fifth order Pell sequence.

Next, we give the ordinary generating function $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Lemma 3.1. *Suppose that $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the generalized fifth order Pell sequence $\{V_n\}_{n\geq 0}$. Then, $f_{b_n}(x)$ is given by*

$$f_{b_n}(x) = \frac{V_0 + (V_1 - 6V_0)x + (11V_0 - 5V_1 + V_2)x^2 + (6V_1 - 9V_0 - 4V_2 + V_3)x^3 + (2V_0 - 3V_1 + 2V_2 - 3V_3 + V_4)x^4}{1 - 7x + 17x^2 - 20x^3 + 11x^4 - 3x^5}. \tag{3.1}$$

Proof. Using Lemma 1.2, we obtain

$$\begin{aligned} f_{b_n}(x) &= \frac{b_0 + (b_1 - rb_0)x + (b_2 - rb_1 - sb_0)x^2 + (b_3 - rb_2 - sb_1 - tb_0)x^3 + (b_4 - rb_3 - sb_2 - tb_1 - ub_0)x^4}{1 - rx - sx^2 - tx^3 - ux^4 - vx^5} \\ &= \frac{V_0 + (V_1 - 6V_0)x + (11V_0 - 5V_1 + V_2)x^2 + (6V_1 - 9V_0 - 4V_2 + V_3)x^3 + (2V_0 - 3V_1 + 2V_2 - 3V_3 + V_4)x^4}{1 - 7x + 17x^2 - 20x^3 + 11x^4 - 3x^5} \end{aligned}$$

where

$$\begin{aligned} b_0 &= V_0, \\ b_1 &= V_0 + V_1, \\ b_2 &= V_0 + 2V_1 + V_2, \\ b_3 &= V_0 + 3V_1 + 3V_2 + V_3, \\ b_4 &= V_0 + 4V_1 + 6V_2 + 4V_3 + V_4. \end{aligned}$$

□

Note that P. Barry shows in [26] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x) = \frac{1}{1-x} A(\frac{x}{1-x})$$

is the generating function of the sequence $\{b_n\}$ with $b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$. In our case, since

$$A(x) = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 + (V_4 - 2V_3 - V_2 - V_1 - V_0)x^4}{(1 - 2x - x^2 - x^3 - x^4 - x^5)},$$

see Lemma 1.7,

we obtain

$$\begin{aligned} S(x) &= \frac{1}{1-x} A(\frac{x}{1-x}) \\ &= \frac{V_0 + (V_1 - 6V_0)x + (11V_0 - 5V_1 + V_2)x^2 + (6V_1 - 9V_0 - 4V_2 + V_3)x^3 + (2V_0 - 3V_1 + 2V_2 - 3V_3 + V_4)x^4}{1 - 7x + 17x^2 - 20x^3 + 11x^4 - 3x^5}. \end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 3.2. *Generating functions of the binomial transform of the fifth order Pell, fifth order Pell-Lucas numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{P}_n x^n &= \frac{x-3x^2+3x^3-x^4}{1-7x+17x^2-20x^3+11x^4-3x^5},\\ \sum_{n=0}^{\infty}\widehat{Q}_n x^n &= \frac{5-28x+51x^2-40x^3+11x^4}{1-7x+17x^2-20x^3+11x^4-3x^5},\end{aligned}$$

respectively.

4 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Pentanacci sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Pentanacci Numbers)**.** *[27, Theorem 3.1] For all integers n, we have*

$$\begin{vmatrix} W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{vmatrix} = v^n \begin{vmatrix} W_4 & W_3 & W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \end{vmatrix}. \tag{4.1}$$

Taking $\{W_n\}=\{b_n\}$ in the above theorem and considering $b_{n+5}=7b_{n+4}-17b_{n+3}+20b_{n+2}-11b_{n+1}+3b_n$, $r=7, s=-17, t=20, u=-11, v=3,$ we have the following proposition.

Proposition 4.1. *For all integers n, Simson formula of binomial transforms of generalized fifth order Pell numbers is given as*

$$\begin{vmatrix} b_{n+4} & b_{n+3} & b_{n+2} & b_{n+1} & b_n \\ b_{n+3} & b_{n+2} & b_{n+1} & b_n & b_{n-1} \\ b_{n+2} & b_{n+1} & b_n & b_{n-1} & b_{n-2} \\ b_{n+1} & b_n & b_{n-1} & b_{n-2} & b_{n-3} \\ b_n & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} \end{vmatrix} = 3^n \begin{vmatrix} b_4 & b_3 & b_2 & b_1 & b_0 \\ b_3 & b_2 & b_1 & b_0 & b_{-1} \\ b_2 & b_1 & b_0 & b_{-1} & b_{-2} \\ b_1 & b_0 & b_{-1} & b_{-2} & b_{-3} \\ b_0 & b_{-1} & b_{-2} & b_{-3} & b_{-4} \end{vmatrix}.$$

The previous proposition gives the following results as particular examples.

Corollary 4.2. *For all integers n, Simson formula of binomial transforms of the fifth order Pell and* *fifth order Pell-Lucas numbers are given as*

$$\begin{vmatrix} \widehat{P}_{n+4} & \widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} \\ \widehat{P}_{n+3} & \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} \\ \widehat{P}_{n+2} & \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} \\ \widehat{P}_{n+1} & \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3} \\ \widehat{P}_{n} & \widehat{P}_{n-1} & \widehat{P}_{n-2} & \widehat{P}_{n-3} & \widehat{P}_{n-4} \end{vmatrix} = 3^{n-4},$$

$$\begin{vmatrix} \widehat{Q}_{n+4} & \widehat{Q}_{n+3} & \widehat{Q}_{n+2} & \widehat{Q}_{n+1} & \widehat{Q}_{n} \\ \widehat{Q}_{n+3} & \widehat{Q}_{n+2} & \widehat{Q}_{n+1} & \widehat{Q}_{n} & \widehat{Q}_{n-1} \\ \widehat{Q}_{n+2} & \widehat{Q}_{n+1} & \widehat{Q}_{n} & \widehat{Q}_{n-1} & \widehat{Q}_{n-2} \\ \widehat{Q}_{n+1} & \widehat{Q}_{n} & \widehat{Q}_{n-1} & \widehat{Q}_{n-2} & \widehat{Q}_{n-3} \\ \widehat{Q}_{n} & \widehat{Q}_{n-1} & \widehat{Q}_{n-2} & \widehat{Q}_{n-3} & \widehat{Q}_{n-4} \end{vmatrix} = 31409 \times 3^{n-4},$$

respectively.

5 SOME IDENTITIES

In this section, we obtain some identities of binomial transforms of generalized fifth order Pell, fifth order Pell and fifth order Pell-Lucas numbers. First, we present a few basic relations between $\{b_n\}$ and $\{\widehat{P}_n\}$.

Lemma 5.1. *The following equalities are true:*

(a) $9b_n = (29b_0 - 254b_1 + 320b_2 - 160b_3 + 25b_4)\widehat{P}_{n+6} - (128b_0 - 1532b_1 + 1994b_2 - 1015b_3 + 160b_4)\widehat{P}_{n+5} + (13b_0 - 2686b_1 + 3772b_2 - 1994b_3 + 320b_4)\widehat{P}_{n+4} + (380b_0 + 1573b_1 - 2686b_2 + 1532b_3 - 254b_4)\widehat{P}_{n+3} - (443b_0 - 380b_1 - 13b_2 + 128b_3 - 29b_4)\widehat{P}_{n+2}$.

(b) $3b_n = (25b_0 - 82b_1 + 82b_2 - 35b_3 + 5b_4)\widehat{P}_{n+5} - (160b_0 - 544b_1 + 556b_2 - 242b_3 + 35b_4)\widehat{P}_{n+4} + (320b_0 - 1169b_1 + 1238b_2 - 556b_3 + 82b_4)\widehat{P}_{n+3} - (254b_0 - 1058b_1 + 1169b_2 - 544b_3 + 82b_4)\widehat{P}_{n+2} + (29b_0 - 254b_1 + 320b_2 - 160b_3 + 25b_4)\widehat{P}_{n+1}$.

(c) $b_n = (5b_0 - 10b_1 + 6b_2 - b_3)\widehat{P}_{n+4} - (35b_0 - 75b_1 + 52b_2 - 13b_3 + b_4)\widehat{P}_{n+3} + (82b_0 - 194b_1 + 157b_2 - 52b_3 + 6b_4)\widehat{P}_{n+2} - (82b_0 - 216b_1 + 194b_2 - 75b_3 + 10b_4)\widehat{P}_{n+1} + (25b_0 - 82b_1 + 82b_2 - 35b_3 + 5b_4)\widehat{P}_n$.

(d) $b_n = (5b_1 - 10b_2 + 6b_3 - b_4)\widehat{P}_{n+3} - (3b_0 + 24b_1 - 55b_2 + 35b_3 - 6b_4)\widehat{P}_{n+2} + (18b_0 + 16b_1 - 74b_2 + 55b_3 - 10b_4)\widehat{P}_{n+1} + (-30b_0 + 28b_1 + 16b_2 - 24b_3 + 5b_4)\widehat{P}_n + 3(5b_0 - 10b_1 + 6b_2 - b_3)\widehat{P}_{n-1}$.

(e) $b_n = -(3b_0 - 11b_1 + 15b_2 - 7b_3 + b_4)\widehat{P}_{n+2} + (18b_0 - 69b_1 + 96b_2 - 47b_3 + 7b_4)\widehat{P}_{n+1} - (30b_0 - 128b_1 + 184b_2 - 96b_3 + 15b_4)\widehat{P}_n + (15b_0 - 85b_1 + 128b_2 - 69b_3 + 11b_4)\widehat{P}_{n-1} + 3(5b_1 - 10b_2 + 6b_3 - b_4)\widehat{P}_{n-2}$.

Proof. Writing

$$b_n = a \times \widehat{P}_{n+6} + b \times \widehat{P}_{n+5} + c \times \widehat{P}_{n+4} + d \times \widehat{P}_{n+3} + e \times \widehat{P}_{n+2}$$

and solving the system of equations

$$\begin{aligned} b_0 &= a \times \widehat{P}_6 + b \times \widehat{P}_5 + c \times \widehat{P}_4 + d \times \widehat{P}_3 + e \times \widehat{P}_2 \\ b_1 &= a \times \widehat{P}_7 + b \times \widehat{P}_6 + c \times \widehat{P}_5 + d \times \widehat{P}_4 + e \times \widehat{P}_3 \\ b_2 &= a \times \widehat{P}_8 + b \times \widehat{P}_7 + c \times \widehat{P}_6 + d \times \widehat{P}_5 + e \times \widehat{P}_4 \\ b_3 &= a \times \widehat{P}_9 + b \times \widehat{P}_8 + c \times \widehat{P}_7 + d \times \widehat{P}_6 + e \times \widehat{P}_5 \\ b_4 &= a \times \widehat{P}_{10} + b \times \widehat{P}_9 + c \times \widehat{P}_8 + d \times \widehat{P}_7 + e \times \widehat{P}_6 \end{aligned}$$

we find that $9a = (29b_0 - 254b_1 + 320b_2 - 160b_3 + 25b_4), 9b = -(128b_0 - 1532b_1 + 1994b_2 - 1015b_3 + 160b_4), 9c = (13b_0 - 2686b_1 + 3772b_2 - 1994b_3 + 320b_4), 9d = (380b_0 + 1573b_1 - 2686b_2 + 1532b_3 -$

$254b_4), 9e = -(443b_0 - 380b_1 - 13b_2 + 128b_3 - 29b_4)$.

The other equalities can be proved similarly. □

Now, we give a few basic relations between $\{b_n\}$ and $\{\widehat{Q}_n\}$.

Lemma 5.2. *The following equalities are true:*

(a) $40383b_n = -(32348b_0 - 150599b_1 + 168251b_2 - 79423b_3 + 12103b_4)\widehat{Q}_{n+6} + (190127b_0 - 953408b_1 + 1086296b_2 - 518461b_3 + 79423b_4)\widehat{Q}_{n+5} - (311647b_0 - 1876657b_1 + 2225215b_2 - 1086296b_3 + 168251b_4)\widehat{Q}_{n+4} + (142207b_0 - 1472866b_1 + 1876657b_2 - 953408b_3 + 150599b_4)\widehat{Q}_{n+3} + (95969b_0 + 142207b_1 - 311647b_2 + 190127b_3 - 32348b_4)\widehat{Q}_{n+2}$.

(b) $13461b_n = -(12103b_0 - 33595b_1 + 30487b_2 - 12500b_3 + 1766b_4)\widehat{Q}_{n+5} + (79423b_0 - 227842b_1 + 211684b_2 - 87965b_3 + 12500b_4)\widehat{Q}_{n+4} - (168251b_0 - 513038b_1 + 496121b_2 - 211684b_3 + 30487b_4)\widehat{Q}_{n+3} + (150599b_0 - 504794b_1 + 513038b_2 - 227842b_3 + 33595b_4)\widehat{Q}_{n+2} - (32348b_0 - 150599b_1 + 168251b_2 - 79423b_3 + 12103b_4)\widehat{Q}_{n+1}$.

(c) $4487b_n = -(1766b_0 - 2441b_1 + 575b_2 + 155b_3 - 46b_4)\widehat{Q}_{n+4} + (12500b_0 - 19359b_1 + 7386b_2 - 272b_3 - 155b_4)\widehat{Q}_{n+3} - (30487b_0 - 55702b_1 + 32234b_2 - 7386b_3 + 575b_4)\widehat{Q}_{n+2} + (33595b_0 - 72982b_1 + 55702b_2 - 19359b_3 + 2441b_4)\widehat{Q}_{n+1} - (12103b_0 - 33595b_1 + 30487b_2 - 12500b_3 + 1766b_4)\widehat{Q}_n$.

(d) $4487b_n = (138b_0 - 2272b_1 + 3361b_2 - 1357b_3 + 167b_4)\widehat{Q}_{n+3} - (465b_0 - 14\,205b_1 + 22\,459b_2 - 10\,021b_3 + 1357b_4)\widehat{Q}_{n+2} - (1725b_0 + 24\,162b_1 - 44\,202b_2 + 22\,459b_3 - 3361b_4)\widehat{Q}_{n+1} + (7323b_0 + 6744b_1 - 24\,162b_2 + 14\,205b_3 - 2272b_4)\widehat{Q}_n - 3(1766b_0 - 2441b_1 + 575b_2 + 155b_3 - 46b_4)\widehat{Q}_{n-1}$.

(e) $4487b_n = (501b_0 - 1699b_1 + 1068b_2 + 522b_3 - 188b_4)\widehat{Q}_{n+2} - (4071b_0 - 14\,462b_1 + 12\,935b_2 - 610b_3 - 522b_4)\widehat{Q}_{n+1} + (10\,083b_0 - 38\,696b_1 + 43\,058b_2 - 12\,935b_3 + 1068b_4)\widehat{Q}_n - (6816b_0 - 32\,315b_1 + 38\,696b_2 - 14\,462b_3 + 1699b_4)\widehat{Q}_{n-1} + 3(138b_0 - 2272b_1 + 3361b_2 - 1357b_3 + 167b_4)\widehat{Q}_{n-2}$.

Next, we present a few basic relations between $\{\widehat{Q}_n\}$ and $\{\widehat{P}_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned}
40383\widehat{P}_n &= -3530\widehat{Q}_{n+6} + 25049\widehat{Q}_{n+5} - 60358\widehat{Q}_{n+4} + 65401\widehat{Q}_{n+3} - 27655\widehat{Q}_{n+2},\\
13461\widehat{P}_n &= 113\widehat{Q}_{n+5} - 116\widehat{Q}_{n+4} - 1733\widehat{Q}_{n+3} + 3725\widehat{Q}_{n+2} - 3530\widehat{Q}_{n+1},\\
4487\widehat{P}_n &= 225\widehat{Q}_{n+4} - 1218\widehat{Q}_{n+3} + 1995\widehat{Q}_{n+2} - 1591\widehat{Q}_{n+1} + 113\widehat{Q}_n,\\
4487\widehat{P}_n &= 357\widehat{Q}_{n+3} - 1830\widehat{Q}_{n+2} + 2909\widehat{Q}_{n+1} - 2362\widehat{Q}_n + 675\widehat{Q}_{n-1},\\
4487\widehat{P}_n &= 669\widehat{Q}_{n+2} - 3160\widehat{Q}_{n+1} + 4778\widehat{Q}_n - 3252\widehat{Q}_{n-1} + 1071\widehat{Q}_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
9\widehat{Q}_n &= -118\widehat{P}_{n+6} + 784\widehat{P}_{n+5} - 1721\widehat{P}_{n+4} + 1691\widehat{P}_{n+3} - 521\widehat{P}_{n+2},\\
3\widehat{Q}_n &= -14\widehat{P}_{n+5} + 95\widehat{P}_{n+4} - 223\widehat{P}_{n+3} + 259\widehat{P}_{n+2} - 118\widehat{P}_{n+1},\\
\widehat{Q}_n &= -\widehat{P}_{n+4} + 5\widehat{P}_{n+3} - 7\widehat{P}_{n+2} + 12\widehat{P}_{n+1} - 14\widehat{P}_n,\\
\widehat{Q}_n &= -2\widehat{P}_{n+3} + 10\widehat{P}_{n+2} - 8\widehat{P}_{n+1} - 3\widehat{P}_n - 3\widehat{P}_{n-1},\\
\widehat{Q}_n &= -4\widehat{P}_{n+2} + 26\widehat{P}_{n+1} - 43\widehat{P}_n + 19\widehat{P}_{n-1} - 6\widehat{P}_{n-2}.
\end{aligned}$$

6 ON THE RECURRENCE PROPERTIES OF BINOMIAL TRANSFORM OF THE GENERALIZED FIFTH ORDER PELL SEQUENCE

Taking $r_1 = 7, s_1 = -17, t_1 = 20, u_1 = -11, v_1 = 3$ and $H_n = \widehat{Q}_n$ in Theorem 1.5, we obtain the following Proposition.

Proposition 6.1. *For $n \in \mathbb{Z}$, binomial Transform of the generalized fifth order Pell sequence have the following identity:*

$b_{-n} = \frac{1}{24}3^{-n}(b_0\widehat{Q}_n^4 - 4b_n\widehat{Q}_n^3 + 3b_0\widehat{Q}_{2n}^2 + 12\widehat{Q}_n^2 b_{2n} - 6b_0\widehat{Q}_n^2\widehat{Q}_{2n} - 6b_0\widehat{Q}_{4n} - 8b_n\widehat{Q}_{3n} - 12\widehat{Q}_{2n}b_{2n} - 24\widehat{Q}_n b_{3n} + 24b_{4n} + 8b_0\widehat{Q}_n\widehat{Q}_{3n} + 12b_n\widehat{Q}_n\widehat{Q}_{2n})$
$= 3^{-n}(b_{4n} - \widehat{Q}_n b_{3n} + \frac{1}{2}(\widehat{Q}_n^2 - \widehat{Q}_{2n})b_{2n} - \frac{1}{6}(\widehat{Q}_n^3 + 2\widehat{Q}_{3n} - 3\widehat{Q}_{2n}\widehat{Q}_n)b_n + \frac{1}{24}(\widehat{Q}_n^4 + 3\widehat{Q}_{2n}^2 - 6\widehat{Q}_n^2\widehat{Q}_{2n} - 6\widehat{Q}_{4n} + 8\widehat{Q}_{3n}\widehat{Q}_n)b_0).$

Using Proposition 6.1 (and Corollary 1.6), we obtain the following corollary which gives the connection between the special cases of binomial transform of generalized fifth order Pell sequence at the positive index and the negative index: for binomial transform of fifth order Pell, fifth order Pell-Lucas numbers: take $b_n = \widehat{P}_n$ with $\widehat{P}_0 = 0, \widehat{P}_1 = 1, \widehat{P}_2 = 4, \widehat{P}_3 = 14, \widehat{P}_4 = 49$, take $b_n = \widehat{Q}_n$ with $\widehat{Q}_0 = 5, \widehat{Q}_1 = 7, \widehat{Q}_2 = 15, \widehat{Q}_3 = 46, \widehat{Q}_4 = 163$, respectively. Note that in this case we have $H_n = \widehat{Q}_n$. Note also that $G_n \neq \widehat{P}_n$.

Corollary 6.1. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Recurrence relations of binomial transforms of fifth order Pell numbers (take $b_n = \widehat{P}_n$ in Proposition 6.1):*

$$\widehat{P}_{-n} = 3^{-n}(\widehat{P}_{4n} - \widehat{Q}_n\widehat{P}_{3n} + \frac{1}{2}(\widehat{Q}_n^2 - \widehat{Q}_{2n})\widehat{P}_{2n} - \frac{1}{6}(\widehat{Q}_n^3 + 2\widehat{Q}_{3n} - 3\widehat{Q}_{2n}\widehat{Q}_n)\widehat{P}_n).$$

(b) *Recurrence relations of binomial transforms of fifth order Pell-Lucas numbers (take $b_n = \widehat{Q}_n$ in Proposition 6.1 or take $H_n = \widehat{Q}_n$ in Corollary 1.6):*

$$\widehat{Q}_{-n} = \frac{1}{24}3^{-n}(\widehat{Q}_n^4 + 3\widehat{Q}_{2n}^2 - 6\widehat{Q}_n^2\widehat{Q}_{2n} - 6\widehat{Q}_{4n} + 8\widehat{Q}_{3n}\widehat{Q}_n).$$

7 SUM FORMULAS

7.1 Sums of Terms with Positive Subscripts

The following proposition presents some formulas of binomial transform of generalized fifth order Pell numbers with positive subscripts.

Proposition 7.1. *If $r = 7, s = -17, t = 20, u = -11, v = 3$, then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} b_k = b_{n+5} - 6b_{n+4} + 11b_{n+3} - 9b_{n+2} + 2b_{n+1} - b_4 + 6b_3 - 11b_2 + 9b_1 - 2b_0.$

(b) $\sum_{k=0}^{n} b_{2k} = \frac{1}{59}(29b_{2n+2} - 173b_{2n+1} + 371b_{2n} - 243b_{2n-1} + 90b_{2n-2} - 29b_4 + 173b_3 - 312b_2 + 243b_1 - 31b_0).$

(c) $\sum_{k=0}^{n} b_{2k+1} = \frac{1}{59}(30b_{2n+2} - 122b_{2n+1} + 337b_{2n} - 229b_{2n-1} + 87b_{2n-2} - 30b_4 + 181b_3 - 337b_2 + 288b_1 - 87b_0).$

Proof. Take $r = 7, s = -17, t = 20, u = -11, v = 3,$ in Theorem 2.1 in [28].

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of fifth order Pell numbers (take $b_n = \widehat{P}_n$ with $\widehat{P}_0 = 0, \widehat{P}_1 = 1, \widehat{P}_2 = 4, \widehat{P}_3 = 14, \widehat{P}_4 = 49$).

Corollary 7.1. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{P}_k = \widehat{P}_{n+5} - 6\widehat{P}_{n+4} + 11\widehat{P}_{n+3} - 9\widehat{P}_{n+2} + 2\widehat{P}_{n+1}.$

(b) $\sum_{k=0}^{n} \widehat{P}_{2k} = \frac{1}{59}(29\widehat{P}_{2n+2} - 173\widehat{P}_{2n+1} + 371\widehat{P}_{2n} - 243\widehat{P}_{2n-1} + 90\widehat{P}_{2n-2} - 4).$

(c) $\sum_{k=0}^{n} \widehat{P}_{2k+1} = \frac{1}{59}(30\widehat{P}_{2n+2} - 122\widehat{P}_{2n+1} + 337\widehat{P}_{2n} - 229\widehat{P}_{2n-1} + 87\widehat{P}_{2n-2} + 4).$

Taking $b_n = \widehat{Q}_n$ with $\widehat{Q}_0 = 5, \widehat{Q}_1 = 7, \widehat{Q}_2 = 15, \widehat{Q}_3 = 46, \widehat{Q}_4 = 163$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of fifth order Pell-Lucas numbers.

Corollary 7.2. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{Q}_k = \widehat{Q}_{n+5} - 6\widehat{Q}_{n+4} + 11\widehat{Q}_{n+3} - 9\widehat{Q}_{n+2} + 2\widehat{Q}_{n+1} + 1.$

(b) $\sum_{k=0}^{n} \widehat{Q}_{2k} = \frac{1}{59}(29\widehat{Q}_{2n+2} - 173\widehat{Q}_{2n+1} + 371\widehat{Q}_{2n} - 243\widehat{Q}_{2n-1} + 90\widehat{Q}_{2n-2} + 97).$

(c) $\sum_{k=0}^{n} \widehat{Q}_{2k+1} = \frac{1}{59}(30\widehat{Q}_{2n+2} - 122\widehat{Q}_{2n+1} + 337\widehat{Q}_{2n} - 229\widehat{Q}_{2n-1} + 87\widehat{Q}_{2n-2} - 38).$

7.2 Sums of Terms with Negative Subscripts

The following proposition presents some formulas of binomial transform of generalized fifth order Pell numbers with negative subscripts.

Proposition 7.2. *If $r = 7, s = -17, t = 20, u = -11, v = 3,$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} b_{-k} = -b_{-n+4} + 6b_{-n+3} - 11b_{-n+2} + 9b_{-n+1} - 2b_{-n} + b_4 - 6b_3 + 11b_2 - 9b_1 + 2b_0.$

(b) $\sum_{k=1}^{n} b_{-2k} = \frac{1}{59}(-30b_{-2n+3} + 181b_{-2n+2} - 337b_{-2n+1} + 288b_{-2n} - 87b_{-2n-1} + 29b_4 - 173b_3 + 312b_2 - 243b_1 + 31b_0).$

(c) $\sum_{k=1}^{n} b_{-2k+1} = \frac{1}{59}(-29b_{-2n+3} + 173b_{-2n+2} - 312b_{-2n+1} + 243b_{-2n} - 90b_{-2n-1} + 30b_4 - 181b_3 + 337b_2 - 288b_1 + 87b_0).$

Proof. Take $r = 7, s = -17, t = 20, u = -11, v = 3,$ in Theorem 3.1 in [28].

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of fifth order Pell numbers (take $b_n = \widehat{P}_n$ with $\widehat{P}_0 = 0, \widehat{P}_1 = 1, \widehat{P}_2 = 4, \widehat{P}_3 = 14, \widehat{P}_4 = 49$).

Corollary 7.3. *For $n \geq 1,$ binomial transform of fifth order Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{P}_{-k} = -\widehat{P}_{-n+4} + 6\widehat{P}_{-n+3} - 11\widehat{P}_{-n+2} + 9\widehat{P}_{-n+1} - 2\widehat{P}_{-n}.$

(b) $\sum_{k=1}^{n} \widehat{P}_{-2k} = \frac{1}{59}(-30\widehat{P}_{-2n+3} + 181\widehat{P}_{-2n+2} - 337\widehat{P}_{-2n+1} + 288\widehat{P}_{-2n} - 87\widehat{P}_{-2n-1} + 4).$

(c) $\sum_{k=1}^{n} \widehat{P}_{-2k+1} = \frac{1}{59}(-29\widehat{P}_{-2n+3} + 173\widehat{P}_{-2n+2} - 312\widehat{P}_{-2n+1} + 243\widehat{P}_{-2n} - 90\widehat{P}_{-2n-1} - 4).$

Taking $b_n = \widehat{Q}_n$ with $\widehat{Q}_0 = 5, \widehat{Q}_1 = 7, \widehat{Q}_2 = 15, \widehat{Q}_3 = 46, \widehat{Q}_4 = 163$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of fifth order Pell-Lucas numbers.

Corollary 7.4. *For $n \geq 1,$ binomial transform of fifth order Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} \widehat{Q}_{-k} = -\widehat{Q}_{-n+4} + 6\widehat{Q}_{-n+3} - 11\widehat{Q}_{-n+2} + 9\widehat{Q}_{-n+1} - 2\widehat{Q}_{-n} - 1.$

(b) $\sum_{k=1}^{n} \widehat{Q}_{-2k} = \frac{1}{59}(-30\widehat{Q}_{-2n+3} + 181\widehat{Q}_{-2n+2} - 337\widehat{Q}_{-2n+1} + 288\widehat{Q}_{-2n} - 87\widehat{Q}_{-2n-1} - 97).$

(c) $\sum_{k=1}^{n} \widehat{Q}_{-2k+1} = \frac{1}{59}(-29\widehat{Q}_{-2n+3} + 173\widehat{Q}_{-2n+2} - 312\widehat{Q}_{-2n+1} + 243\widehat{Q}_{-2n} - 90\widehat{Q}_{-2n-1} + 38).$

8 MATRICES RELATED WITH BINOMIAL TRANSFORM OF GENERALIZED FIFTH ORDER PELL NUMBERS

We define the square matrix A of order 5 as:

$$A=\begin{pmatrix} 7 & -17 & 20 & -11 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A=3$. From (1.1) we have

$$\begin{pmatrix} b_{n+4} \\ b_{n+3} \\ b_{n+2} \\ b_{n+1} \\ b_{n} \end{pmatrix}=\begin{pmatrix} 7 & -17 & 20 & -11 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} b_{n+3} \\ b_{n+2} \\ b_{n+1} \\ b_{n} \\ b_{n-1} \end{pmatrix}. \tag{8.1}$$

and from (1.6) (or using (8.1) and induction) we have

$$\begin{pmatrix} b_{n+4} \\ b_{n+3} \\ b_{n+2} \\ b_{n+1} \\ b_{n} \end{pmatrix}=\begin{pmatrix} 7 & -17 & 20 & -11 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^{n}\begin{pmatrix} b_{4} \\ b_{3} \\ b_{2} \\ b_{1} \\ b_{0} \end{pmatrix}.$$

If we take $b_n=\widehat{P}_n$ in (8.1) we have

$$\begin{pmatrix} \widehat{P}_{n+4} \\ \widehat{P}_{n+3} \\ \widehat{P}_{n+2} \\ \widehat{P}_{n+1} \\ \widehat{P}_{n} \end{pmatrix}=\begin{pmatrix} 7 & -17 & 20 & -11 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \widehat{P}_{n+3} \\ \widehat{P}_{n+2} \\ \widehat{P}_{n+1} \\ \widehat{P}_{n} \\ \widehat{P}_{n-1} \end{pmatrix}. \tag{8.2}$$

We also, for $n\geq 0$, define

$$B_n=\begin{pmatrix} \sum_{k=0}^{n+1}\sum_{l=k}^{n+1}\sum_{p=l}^{n+1}\widehat{P}_k & E_1 & E_6 & E_{11} & 3\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{P}_k \\ \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{P}_k & E_2 & E_{11} & E_{12} & 3\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k \\ \sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k & E_3 & E_8 & E_{13} & 3\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k \\ \sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k & E_4 & E_9 & E_{14} & 3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k \\ \sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k & E_5 & E_{10} & E_{15} & 3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k \end{pmatrix}$$

and

$$C_n=\begin{pmatrix} b_{n+1} & -17b_n+20b_{n-1}-11b_{n-2}+3b_{n-3} & 20b_n-11b_{n-1}+3b_{n-2} & -11b_n+3b_{n-1} & 3b_n \\ b_n & -17b_{n-1}+20b_{n-2}-11b_{n-3}+3b_{n-4} & 20b_{n-1}-11b_{n-2}+3b_{n-3} & -11b_{n-1}+3b_{n-2} & 3b_{n-1} \\ b_{n-1} & -17b_{n-2}+20b_{n-3}-11b_{n-4}+3b_{n-5} & 20b_{n-2}-11b_{n-3}+3b_{n-4} & -11b_{n-2}+3b_{n-3} & 3b_{n-2} \\ b_{n-2} & -17b_{n-3}+20b_{n-4}-11b_{n-5}+3b_{n-6} & 20b_{n-3}-11b_{n-4}+3b_{n-5} & -11b_{n-3}+3b_{n-4} & 3b_{n-3} \\ b_{n-3} & -17b_{n-4}+20b_{n-5}-11b_{n-6}+3b_{n-11} & 20b_{n-4}-11b_{n-5}+3b_{n-6} & -11b_{n-4}+3b_{n-5} & 3b_{n-4} \end{pmatrix}$$

where

$$\begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{matrix} = \begin{matrix} -17\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{P}_k+20\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k-11\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k+3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k \\ -17\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k+20\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k-11\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k+3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k \\ -17\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k+20\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k-11\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k+3\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{P}_k \\ -17\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k+20\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k-11\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{P}_k+3\sum_{k=0}^{n-6}\sum_{l=k}^{n-6}\sum_{p=l}^{n-6}\widehat{P}_k \\ -17\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k+20\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{P}_k-11\sum_{k=0}^{n-6}\sum_{l=k}^{n-6}\sum_{p=l}^{n-6}\widehat{P}_k+3\sum_{k=0}^{n-7}\sum_{l=k}^{n-7}\sum_{p=l}^{n-7}\widehat{P}_k \end{matrix}$$

and

$$\begin{array}{ccc} E_6 & & 20\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{P}_k-7\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k+3\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k \\ E_7 & & 20\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k-7\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k+3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k \\ E_8 & = & 20\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k-7\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k+3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k \\ E_9 & & 20\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k-7\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k+3\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{P}_k \\ E_{10} & & 20\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k-7\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{P}_k+3\sum_{k=0}^{n-6}\sum_{l=k}^{n-6}\sum_{p=l}^{n-6}\widehat{P}_k \end{array}$$

and

$$\begin{array}{ccc} E_{11} & & -11\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{p=l}^{n}\widehat{P}_k+3\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k \\ E_{12} & & -11\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\sum_{p=l}^{n-1}\widehat{P}_k+3\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k \\ E_{13} & = & -11\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\sum_{p=l}^{n-2}\widehat{P}_k+3\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k \\ E_{14} & & -11\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\sum_{p=l}^{n-3}\widehat{P}_k+3\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k \\ E_{15} & & -11\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\sum_{p=l}^{n-4}\widehat{P}_k+3\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\sum_{p=l}^{n-5}\widehat{P}_k \end{array}$$

By convention, we assume that

$$\begin{aligned} \sum_{k=0}^{0}\sum_{l=k}^{0}\sum_{p=l}^{0}\widehat{P}_k &= 0,\sum_{k=0}^{-1}\sum_{l=k}^{-1}\sum_{p=l}^{-1}\widehat{P}_k=0,\sum_{k=0}^{-2}\sum_{l=k}^{-2}\sum_{p=l}^{-2}\widehat{P}_k=0, \\ \sum_{k=0}^{-3}\sum_{l=k}^{-3}\sum_{p=l}^{-3}\widehat{P}_k &= 0,\sum_{k=0}^{-4}\sum_{l=k}^{-4}\sum_{p=l}^{-4}\widehat{P}_k=\frac{1}{3},\sum_{k=0}^{-5}\sum_{l=k}^{-5}\sum_{p=l}^{-5}\widehat{P}_k=\frac{11}{9}, \\ \sum_{k=0}^{-6}\sum_{l=k}^{-6}\sum_{p=l}^{-6}\widehat{P}_k &= \frac{61}{27},\sum_{k=0}^{-7}\sum_{l=k}^{-7}\sum_{p=l}^{-7}\widehat{P}_k=\frac{164}{81}. \end{aligned}$$

Theorem 8.1. *For all integers* $m,n\geq 0$, *we have*

(a) $B_n = A^n$.

(b) $C_1A^n = A^nC_1$.

(c) $C_{n+m} = C_nB_m = B_mC_n$.

Proof.

(a) Proof can be done by mathematical induction on n.

(b) After matrix multiplication, (b) follows.

(c) We have $C_n = AC_{n-1}$. From the last equation, using induction, we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

□

Theorem 8.2. *For $m, n \geq 0,$ we have*

$$\begin{aligned} b_{n+m} &= b_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1}\widehat{P}_k \\ &+b_{n-1}(-17\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k+20\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k-11\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{P}_k+3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{P}_k) \\ &+b_{n-2}(20\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k-11\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k+3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{P}_k) \\ &+b_{n-3}(-11\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k+3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k)+3b_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k. \end{aligned}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n,$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof. □

Corollary 8.3. *For $m, n \geq 0,$ we have*

$$\begin{aligned} \widehat{P}_{n+m} &= \widehat{P}_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1}\widehat{P}_k \\ &+\widehat{P}_{n-1}(-17\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k+20\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k-11\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{P}_k+3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{P}_k) \\ &+\widehat{P}_{n-2}(20\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k-11\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k+3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{P}_k) \\ &+\widehat{P}_{n-3}(-11\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k+3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k)+3\widehat{P}_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k \end{aligned}$$

and

$$\begin{aligned} \widehat{Q}_{n+m} &= \widehat{Q}_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1}\sum_{p=l}^{m+1}\widehat{P}_k \\ &+\widehat{Q}_{n-1}(-17\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k+20\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k-11\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{P}_k+3\sum_{k=0}^{m-3}\sum_{l=k}^{m-3}\sum_{p=l}^{m-3}\widehat{P}_k) \\ &+\widehat{Q}_{n-2}(20\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k-11\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k+3\sum_{k=0}^{m-2}\sum_{l=k}^{m-2}\sum_{p=l}^{m-2}\widehat{P}_k) \\ &+\widehat{Q}_{n-3}(-11\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k+3\sum_{k=0}^{m-1}\sum_{l=k}^{m-1}\sum_{p=l}^{m-1}\widehat{P}_k)+3\widehat{Q}_{n-4}\sum_{k=0}^{m}\sum_{l=k}^{m}\sum_{p=l}^{m}\widehat{P}_k. \end{aligned}$$

9 CONCLUSIONS

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduced the binomial transform of the generalized fifth order Pell sequence and as special cases, the binomial transform of the fifth order Pell and fifth order Pell-Lucas sequences has been defined. For applications of binomial transform, one can consult the on-line encyclopedia of integer sequences [29]. Just search for "applications of binomial transform" and follow the links provided.

- In section 1, we present some background about the generalized 5-step Fibonacci numbers (also called the generalized Pentanacci numbers).
- In section 2, we define the binomial transform of the generalized fifth order Pell sequence.
- In section 3, we give Binet's formulas and generating functions of the binomial transform of the generalized fifth order Pell sequence.
- In section 4, we present Simson formulas of the binomial transform of the generalized fifth order Pell sequence.
- In section 5, we obtaine some identities of the binomial transform of the generalized fifth order Pell sequence.
- In section 6, we present recurrence relations of binomial transforms of generalized fifth order Pell numbers
- In section 7, we present sum formulas of the binomial transform of the generalized fifth order Pell sequence.
- In section 8, we give some matrix formulation of the binomial transform of the generalized fifth order Pell sequence.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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